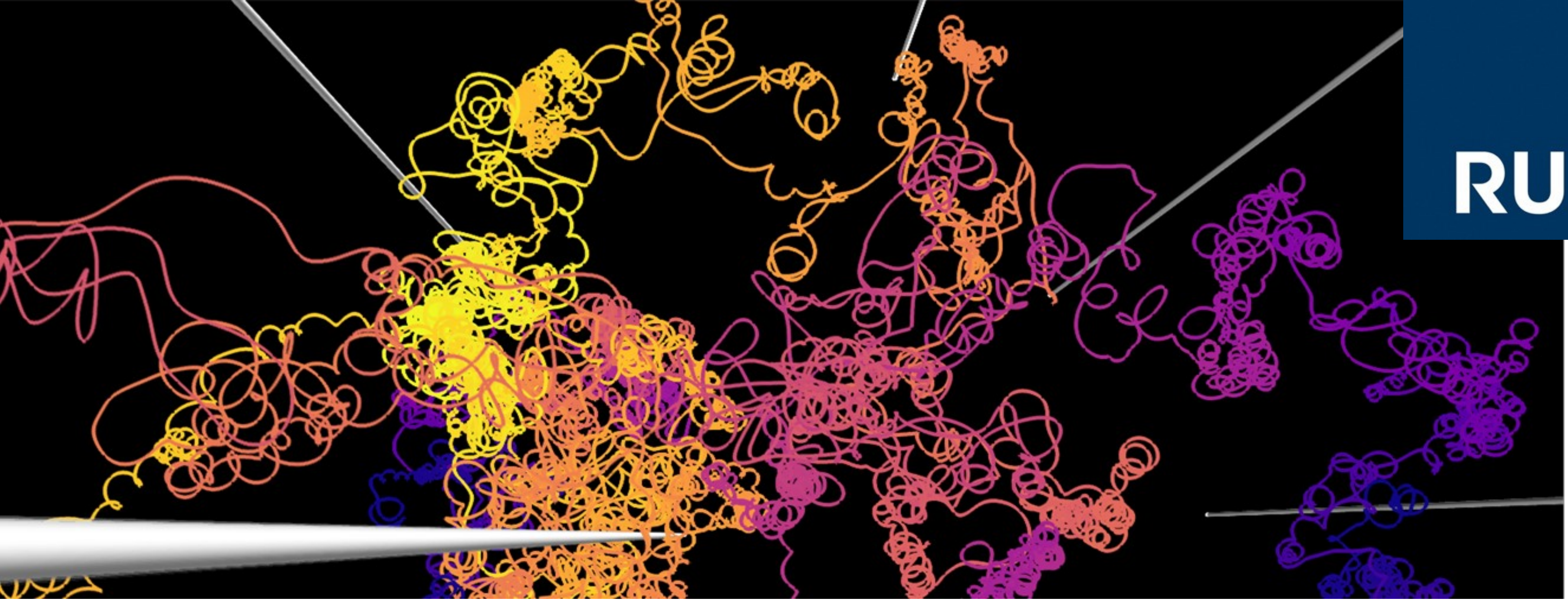

CRPropa's SDE Solver – Connecting Diffusion Tensor Models with efficient CR Transport

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Contents

Ensemble Averaged Transport

Ensemble Averaged Transport

- Time Evolution of **distribution function**

- Information about the **particle ensemble** but not individual CRs

- Abstract quantities, e.g., **diffusion tensor** need to be provided

Parker Transport Equation

$$\frac{\partial n(\vec{r},p,t)}{\partial t} + \underbrace{\boxed{\vec{u}\cdot\nabla n}}_{\text{Advection}} = \overbrace{\boxed{\nabla\cdot(\hat{\kappa}\nabla n)}}^{\text{Spatial Diffusion}} + \underbrace{\boxed{\frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2\kappa_{pp}\frac{\partial n}{\partial p}\right)}}_{\text{Momentum diffusion}} + \overbrace{\boxed{\frac{p}{3}(\nabla\cdot\vec{u})\frac{\partial n}{\partial p}}}^{\text{Adiabatic Effects}} + \underbrace{\boxed{S}}_{\text{Sources}}$$

Stochastic Differential Equations

Stochastic Differential Equation

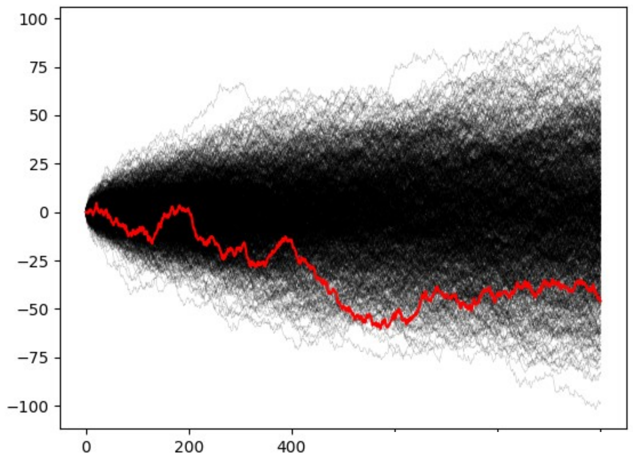
$$\mathrm{d}x = \overbrace{A(x,t)\,\mathrm{d}t}^{\text{Deterministic}} + \underbrace{B(x,t)\,\mathrm{d}W_t}_{\text{Stochastic}}$$

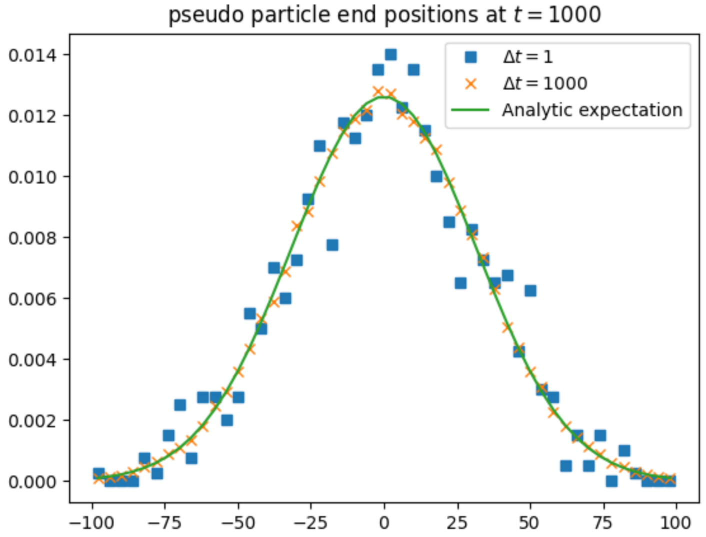
Wiener Process W_t

Properties

1. $W_0 = W(t = 0) = 0$
2. W is time continuous
3. For $s < t \in [0, \infty) \rightarrow W_t - W_s \propto N(0, t - s)$
4. For any $0 \leq s \leq t \leq u < v \in [0, \infty) \rightarrow W_t - W_s$ and $W_v - W_u$ are independent

Trajectories of the standard Wiener process



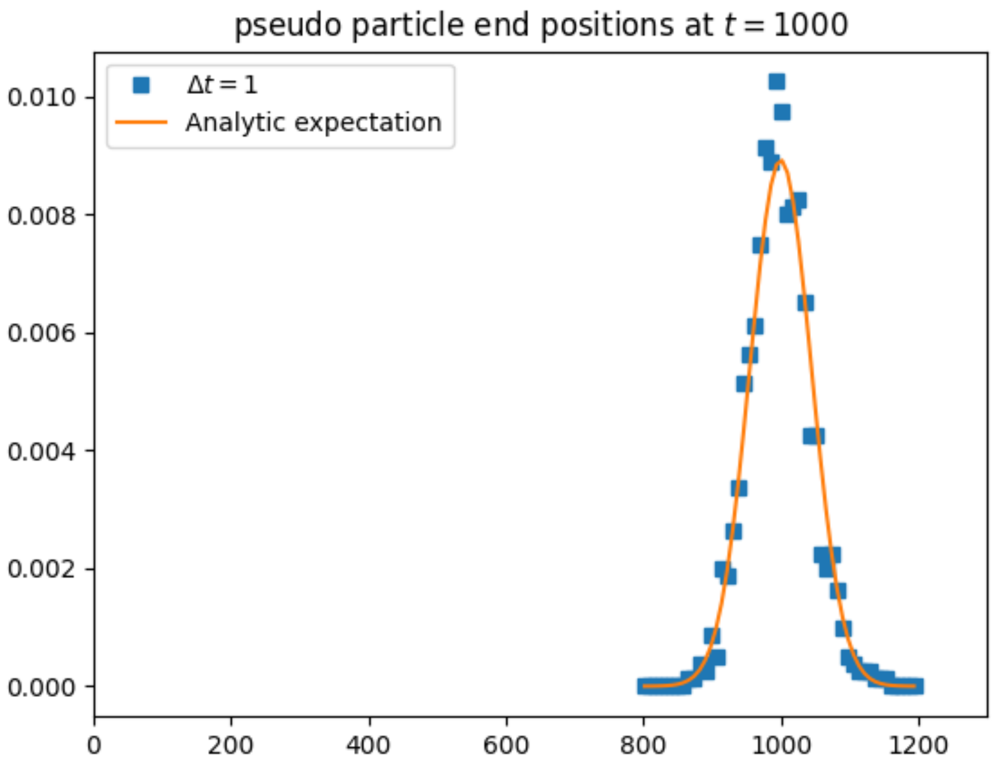


Example – One Dimensional SDE

Drift term and constant diffusion

- $\mathrm{d}x = A(x,t)\mathrm{d}t + B(x,t)\mathrm{d}W_t$
- $A = 1, B = 1$
- $x(t = 0) = 0$
- $T_{\mathrm{max}} = 1000$

Using SDEs to solve a
1-dimensional diffusion advection equation



From Fokker Planck to Stochastic Differential Equation

Fokker Planck Equation:

$$\frac{\partial f(x,t)}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}B(x,t)\right]f(x,t)$$

Ito's Lemma

Stochastic Differential Equation:

$$\mathrm{d}x = A'(x,t)\mathrm{d}t + B'(x,t)\mathrm{d}W_t$$

Current Implementation

CRPropa – Cosmic Ray Propagation Framework

- Monte Carlo simulation framework

Extra-galactic UHECR

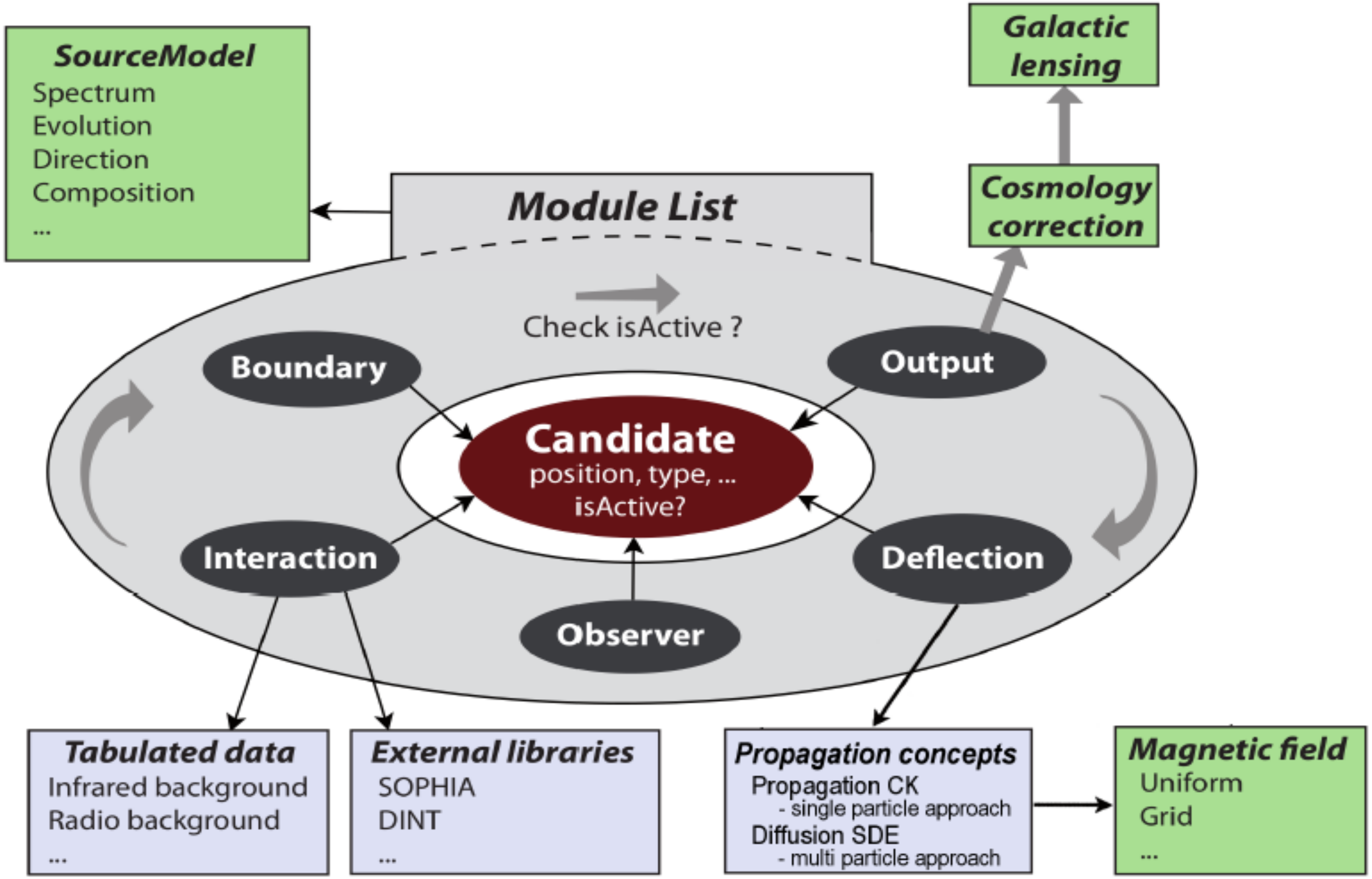
- Models individual CRs or phase-space elements

in source propagation/
Galactic CR

- Includes many of the relevant interactions
- Open Source and modular structure

LM et al. JCAP (2017)

Basic Principle – The Candidate



How to build a simulation?

- SimplePropagation
- PropagationCK
- PropagationBP
- DiffusionSDE

Deflection

- ElectronPairProduction
- PhotoPionProduction
- PhotoDisintegration
- NuclearDecay

Nuclei-Interacion

- EM(Double/Triple)-PairProduction
- EMInverseCompton-Scattering

EM-Interactions

- Redshift
- SynchrotronRadiation
- AdiabaticCooling

General Interactions

- MaximumTrajectory-Length
- MinimumEnergy
- CubicBoundary
- SphericalBoundary
- …

Boundaries/ Thresholds

- ObserverSmallSphere
- ObserverTracking
- ObserverPoint
- ObserverDetectAll
- ObserverTime-Evolution

Observer

- ShellOutput
- TextOutput
- HDF5Output
- ParticleCollector

Output

- PerformanceModule

Others

DiffusionSDE Module

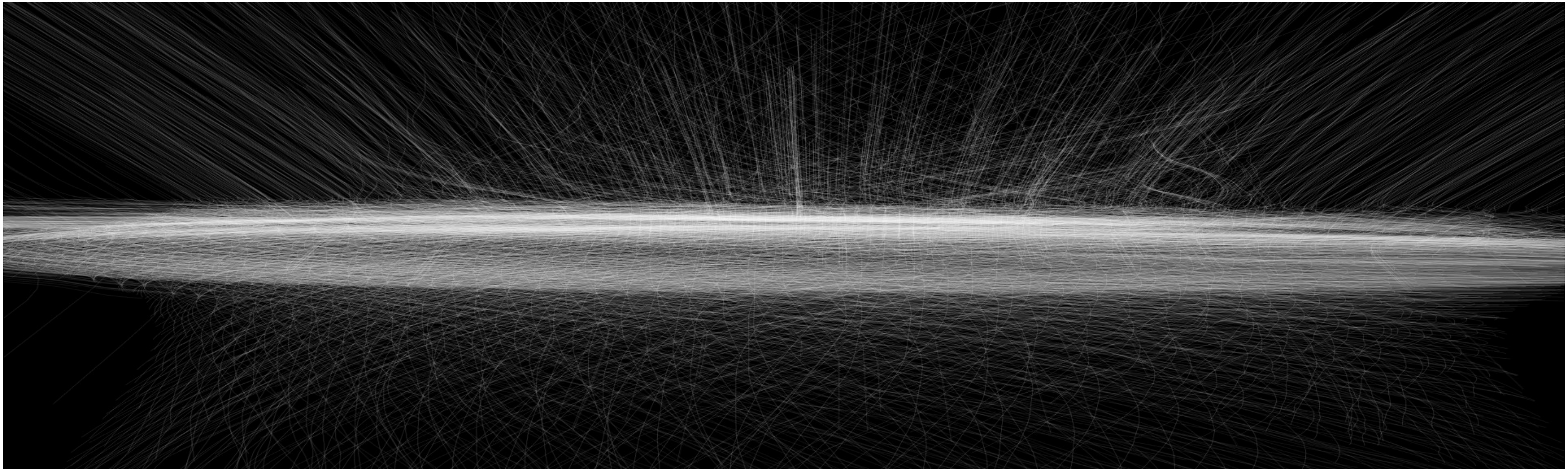
- Arbitrary background magnetic fields
- Anisotropic transport w.r.t. the background field

- Spatially constant Eigenvalues of diffusion tensor $\kappa_\parallel$ and $\kappa_\perp$
- Energy dependence is a single power law
- Ultra-Relativistic particles: $v = c_0$

$$\boxed{\begin{aligned}\vec{x}_{n+1} - \vec{x}_n = {} & (u_x \vec{e}_x + u_y \vec{e}_y + u_z \vec{e}_z) \cdot h \\ & + \left(\sqrt{2\kappa_\parallel}\eta_\parallel \vec{e}_\mathrm{t} + \sqrt{2\kappa_\perp}\eta_{\perp,1}\vec{e}_\mathrm{n} + \sqrt{2\kappa_\perp}\eta_{\perp,2}\vec{e}_\mathrm{b}\right) \cdot \sqrt{h}\end{aligned}}$$

Advantages

- Uses a simple algorithm for the SDE
- No transformation needed by the user
- Works in complicated background fields



Disadvantages

- A lot of code for calculation of local trihedron
  - Computation overhead in simple fields
  - Hard to maintain

- Space dependent Eigenvalues not available
  - No sophisticated shock models
  - No drift terms

- Not easy to extend to anomalous diffusion

Variable Diffusion Tensors

From FP to SDE

Spatial Diffusion

Adiabatic Effects

$$\frac{\partial n(\vec{r},p,t)}{\partial t} + \boxed{\vec{u}\cdot\nabla n} = \boxed{\nabla\cdot(\hat{\kappa}\nabla n)} + \boxed{\frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2\kappa_{pp}\frac{\partial n}{\partial p}\right)} + \boxed{\frac{p}{3}(\nabla\cdot\vec{u})\frac{\partial n}{\partial p}} + \boxed{S}$$

Advection

Momentum diffusion

Sources

Equivalence?

$$\mathrm{d}\vec{x} = \underline{\vec{u}dt} + \underline{\hat{D}\mathrm{d}\vec{w}}$$

$$\mathrm{d}p = \underline{-{}^{p}/_{3}\,(\nabla\cdot\vec{u})\mathrm{d}t} + \underline{D_{pp}\mathrm{d}w_p}$$

Fokker Planck rewrite of Parker Transport Equ.

$$\frac{\partial n}{\partial t} = \nabla\left((\hat{\kappa}\nabla n - \vec{v})n\right) - \vec{w}\,\nabla n + p^3\nabla\vec{w}\,\frac{\partial}{\partial p}\left(\frac{n}{p^2}\right) + \frac{\partial}{\partial p}\left(D_{pp}p^2\frac{\partial}{\partial p}\left(\frac{n}{p^2}\right)\right) - Ln + S$$

$$\begin{aligned}\frac{\partial n}{\partial t} &= \frac{1}{2}\nabla^2(2\hat{\kappa}n) - \nabla\left((\nabla\kappa + \vec{u})n\right)\\ &+\frac{1}{2}\frac{\partial^2}{\partial p^2}(2Dn) - \frac{\partial}{\partial p}\left(\left(\frac{2}{p}D + \frac{\partial D}{\partial p} - \frac{\mathrm{p}}{3}\nabla\vec{w}\right)n\right)\\ &-Ln\\ &+\mathrm{S}\end{aligned}$$

What about terms linear in f(x, t)?

$$\frac{\partial f(x,t)}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}B(x,t)\right]f(x,t) - \mathrm{C(x,t)f(x,t)}$$

Usually included in CR transport equations.

→ Transformation needs to be adapted.

→Apply weights to the phase space element (*Candidate*)

$$w_0 = \exp(-C(x,t)\mathrm{d}\,t)$$

What about constant terms?

$$\frac{\partial f(x,t)}{\partial t} = \left[-\frac{\partial}{\partial x} A(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2} B(x,t)\right] f(x,t) - \mathrm{C(x,t)f(x,t)} + \mathrm{D(x,t)}$$

Usually included in CR transport equations, e.g., source terms

→Apply additional set of weights to the phase space element (*Candidate*)

$$w_1 = \mathrm{const.}$$

Exact form depends on the setting and introduces physics units, too.

Momentum Diffusion

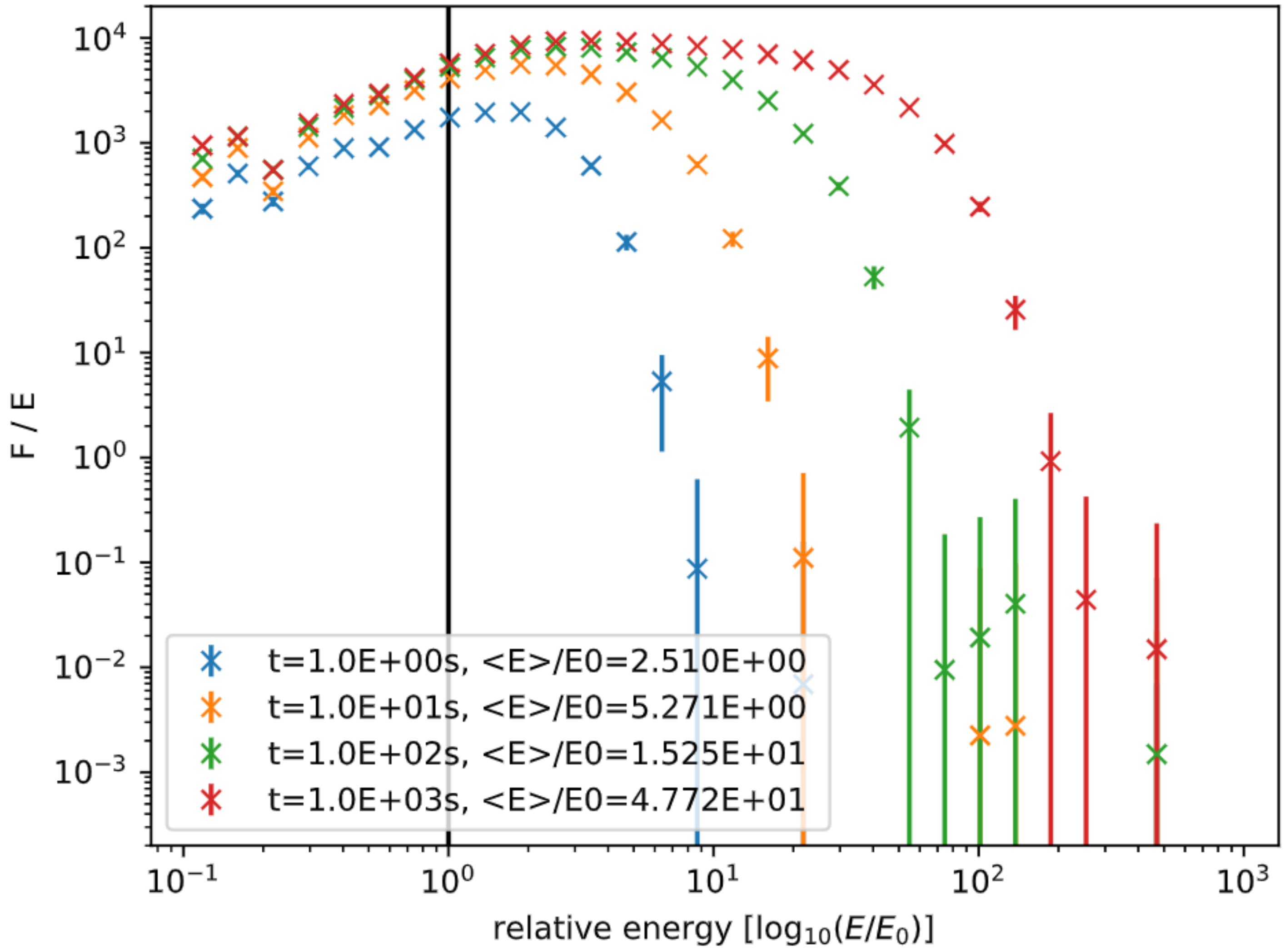
Momentum Diffusion

- Second order Fermi Acceleration
  - Can lead to harder spectra
  - Usually slower than DAS and more efficient at low energies

- Works for constant momentum diffusion coefficient $D_{pp} = \mathrm{const.}$

- Additional Drift terms **not yet** included

Example – Constant Momentum Diffusion



Summary and Outlook

Summary

- Parker Transport Equation **is not** a Fokker Planck Equation
  - → Has to be rewritten

- SDE becomes more complicated:
  - Drifts
  - Weights

- Current implementation is optimized for
  - Constant diffusion tensors
  - Complex background fields

Outlook – General SDE Solver

- Re-Usable, e.g., to solve for distribution function instead of density
- Starting from *time forward* SDE
- Unified handling of spatial and momentum diffusion
- Allow for different SDE solver, e.g., second order schemes
  - Might be needed for drifts at shock fronts

Future Plans

- Implementation and Testing of generalized approach
  - Revise DAS with spatially varying κ
- Examine super- and potentially sub-diffusion in more complex settings
  - See Sophie's talk

Discussion on Design and Requirements of the general SDE Solver

Backup

Examples – Transport in the Milky Way

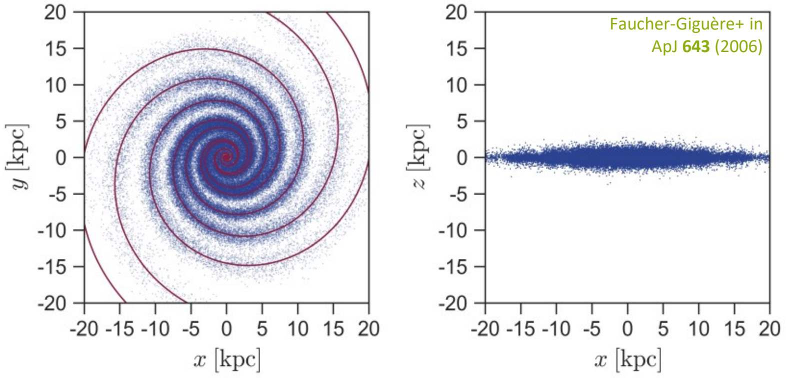
- Source distribution or magnetic field morphology?
- Influence of diffusion ratio ϵ on escape time scales?
- "Classical" scaling of the escape time scale $\tau_{\mathrm{esc}} \propto \kappa^{-1}$?

Source Distribution

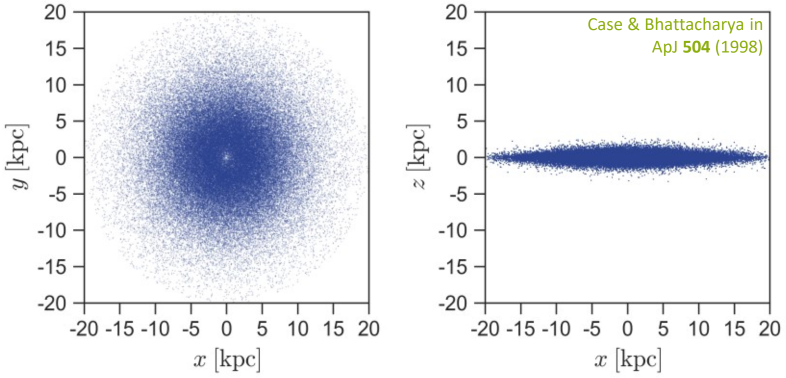
Compare different source distributions with each other

- Older simulation often assumed a homogeneous cylinder
- Likely source classes (supernova remnants, pulsar wind nebulae, etc.) have a spatial structure
- Burst injection: $S \propto \delta(t - t_0)$
- Injected energies: $E/Z = R \in (10 - 10^5)\ \mathrm{TV}$

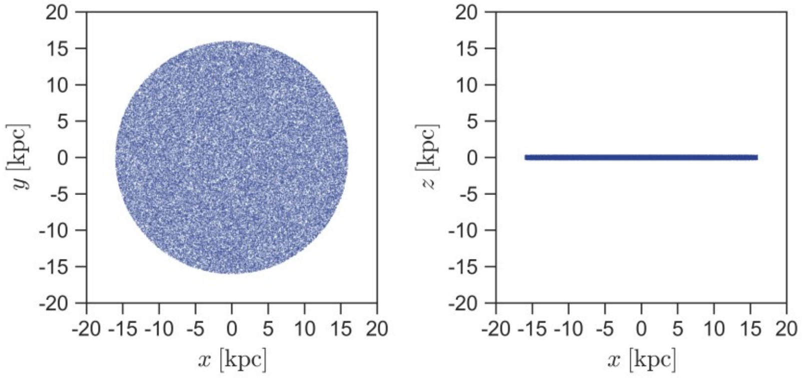
Pulsar Distribution



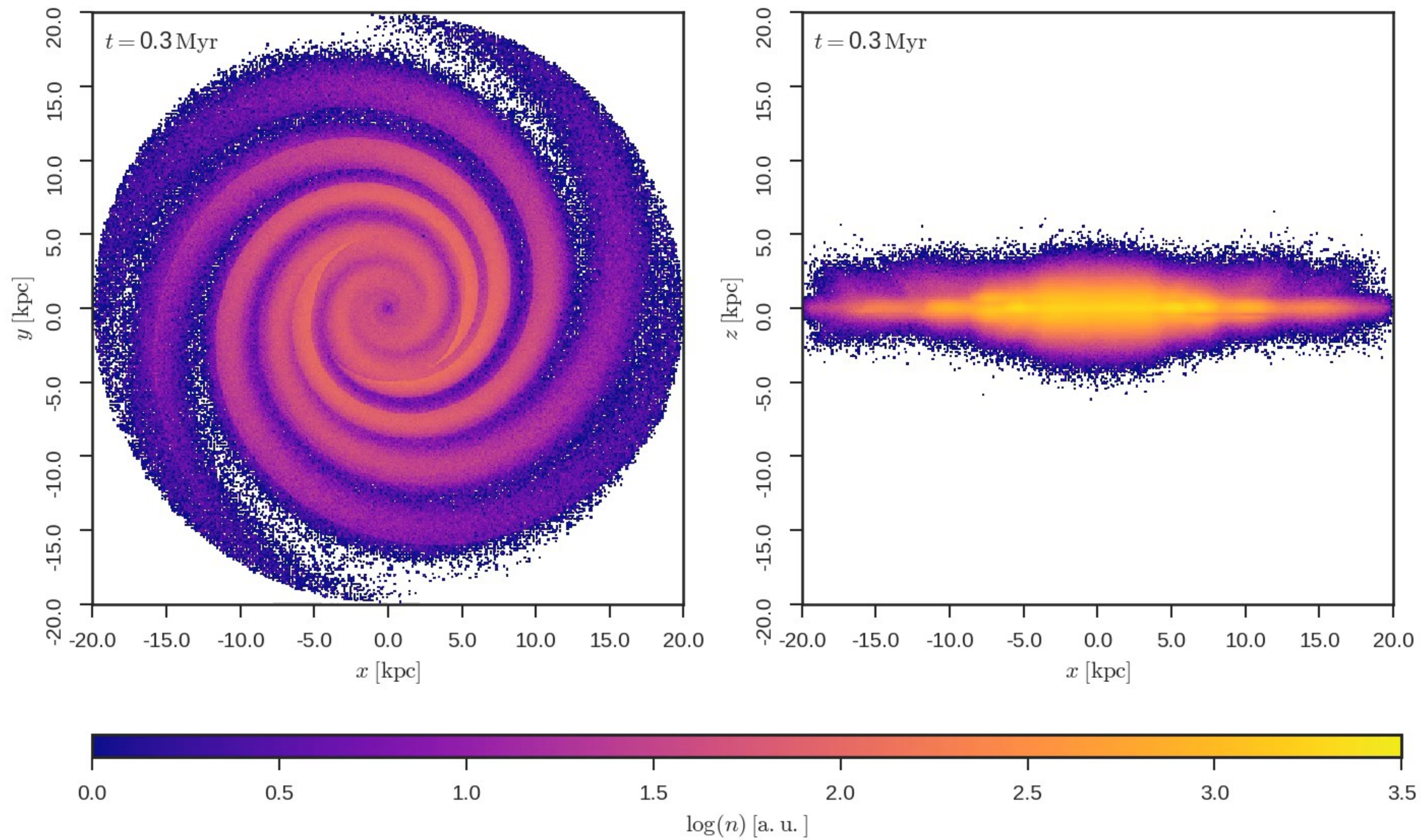
SNR Distribution



Homogeneous Cylinder



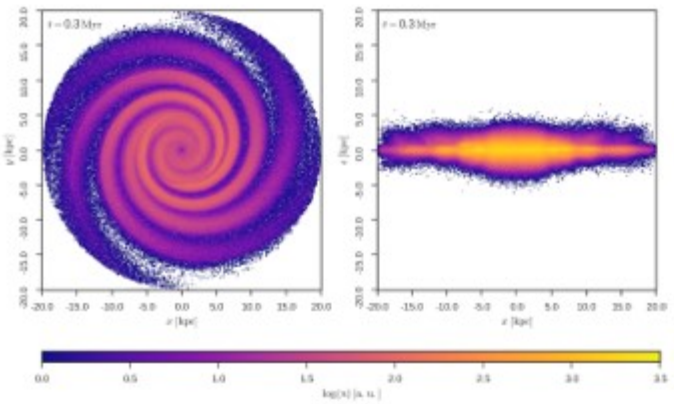
Time Evolution



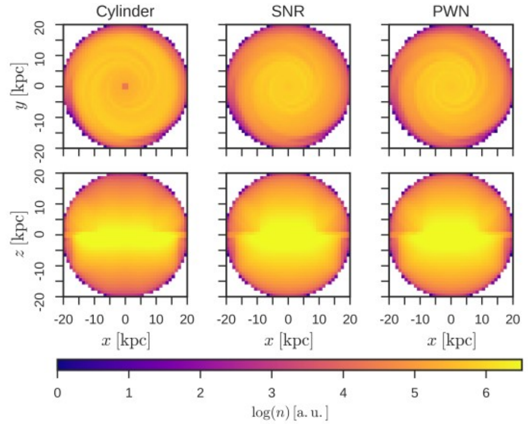
Summary

- Source distribution is relevant on short timescales only.
- Magnetic field morphology plays an important role for the stationary CR distribution.
- Diffusion ratio determines the magnetic field‘s influence on CR density.
- Time scales are decreasing with increasing rigidity.

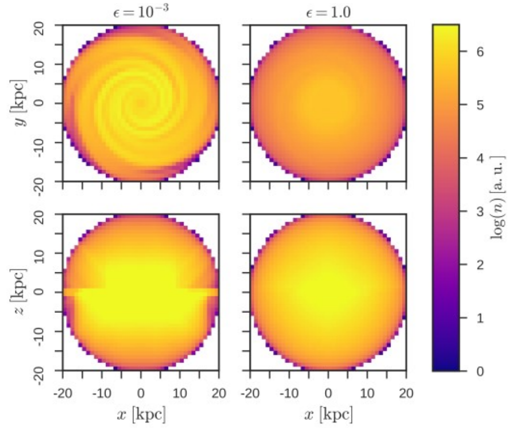
Source: PWN, $R = 10$ TV, $\epsilon = 0.01$



$R = 100$ TV, $\epsilon = 0.01$

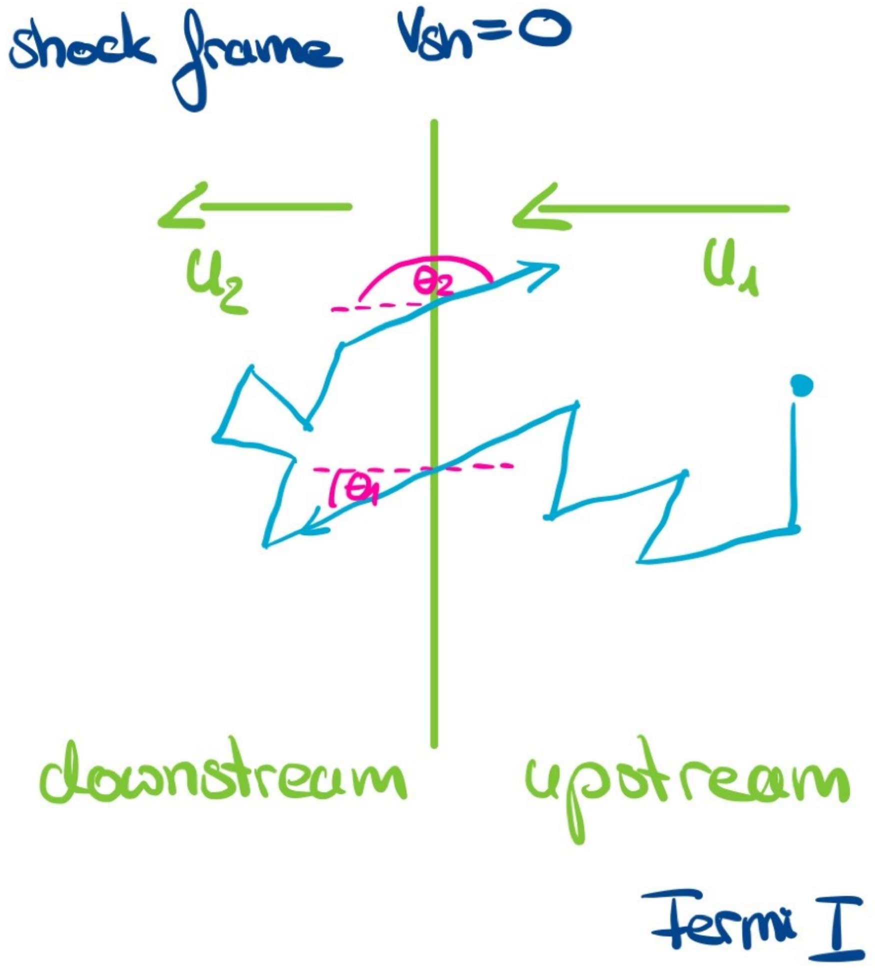


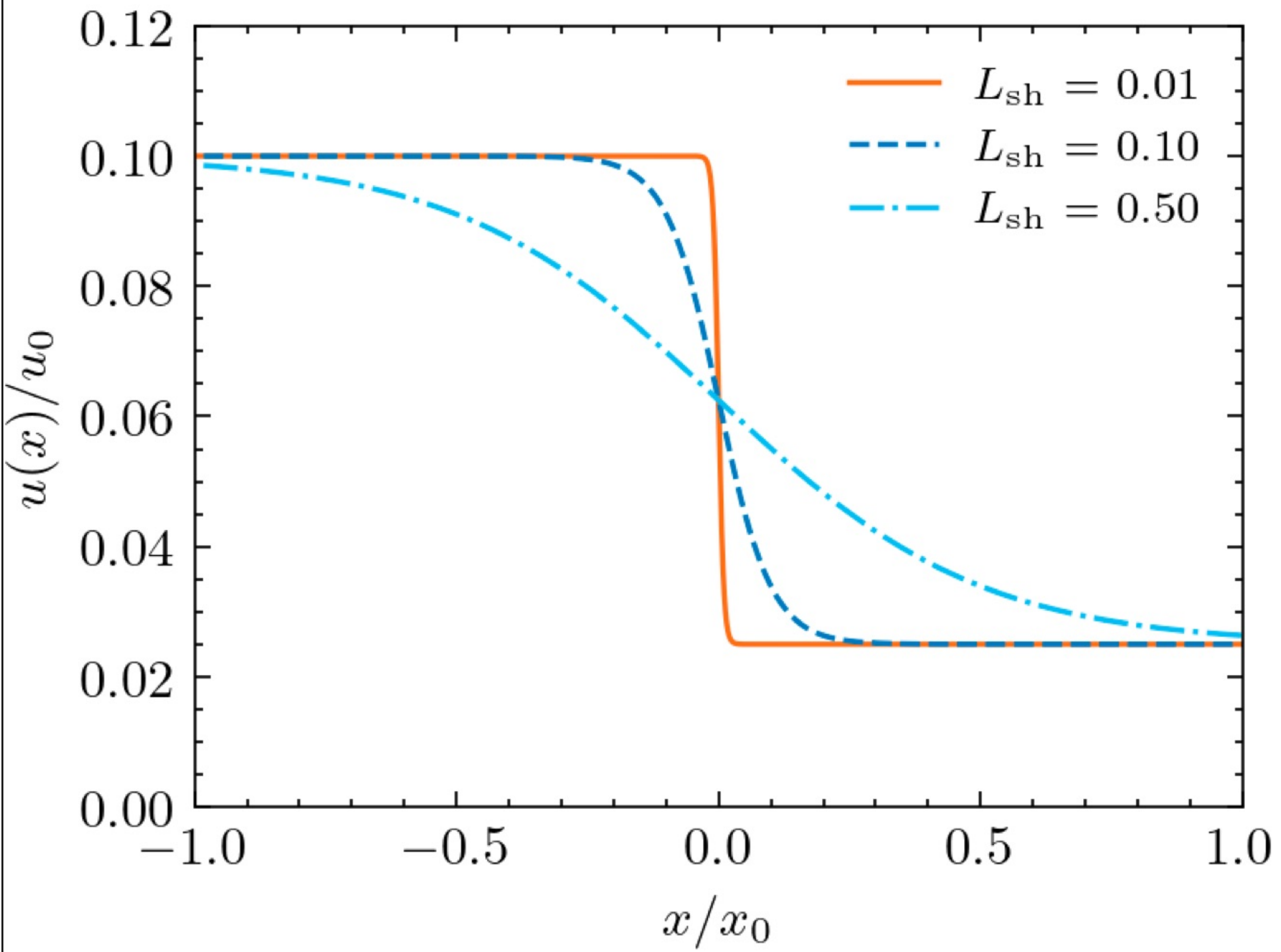
Source: PWN, $R = 10$ TV



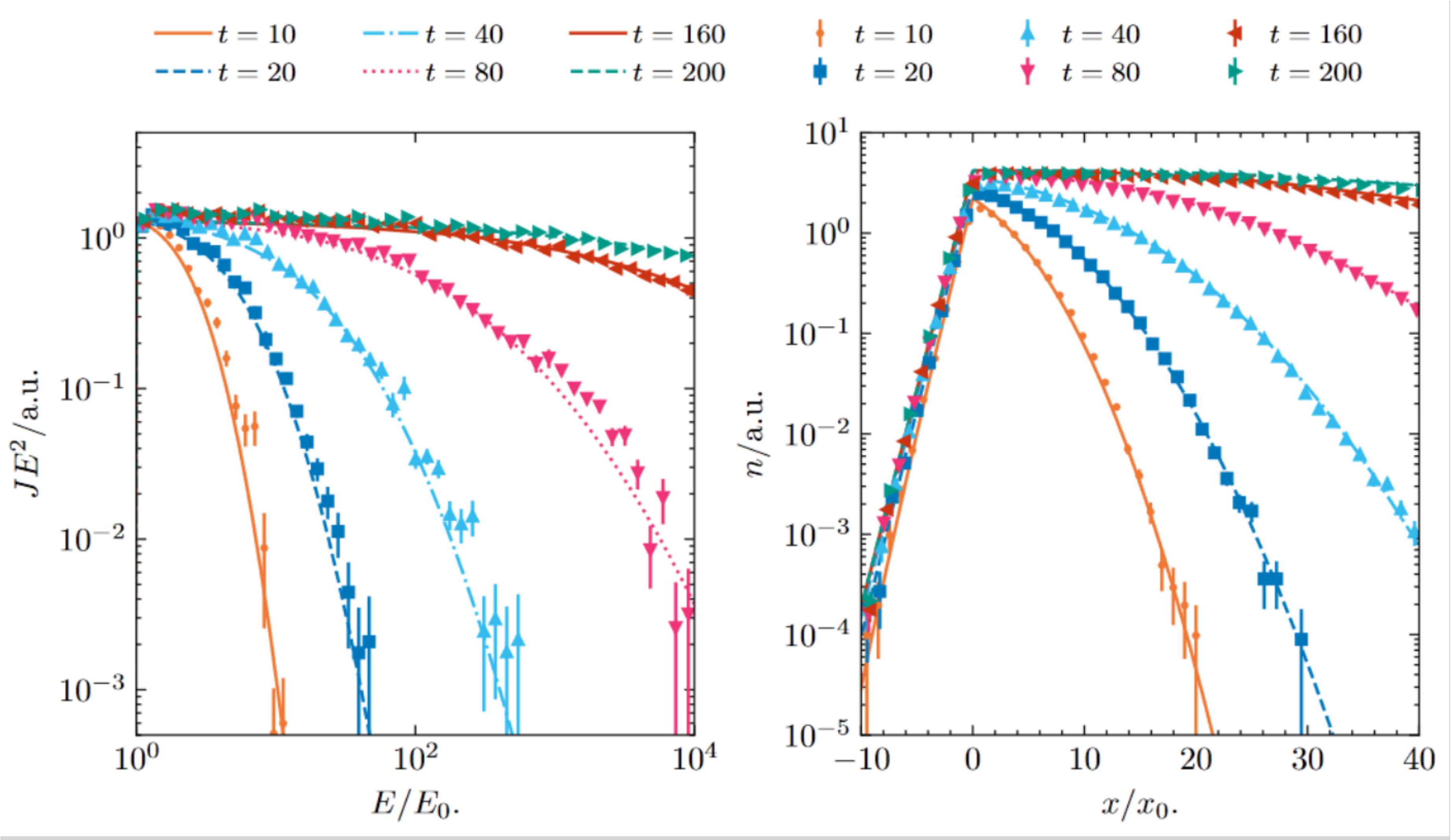
Escape time depends on diffusion ratio and rigidity $\tau_{\mathrm{esc}} = (53 \pm 4) \cdot \epsilon^{-0{,}102 \pm 0{,}016} \cdot (R/_{10\,\mathrm{TV}})^{0{,}30 \pm 0{,}02}$ Myr

Examples – Diffusive Shock Acceleration

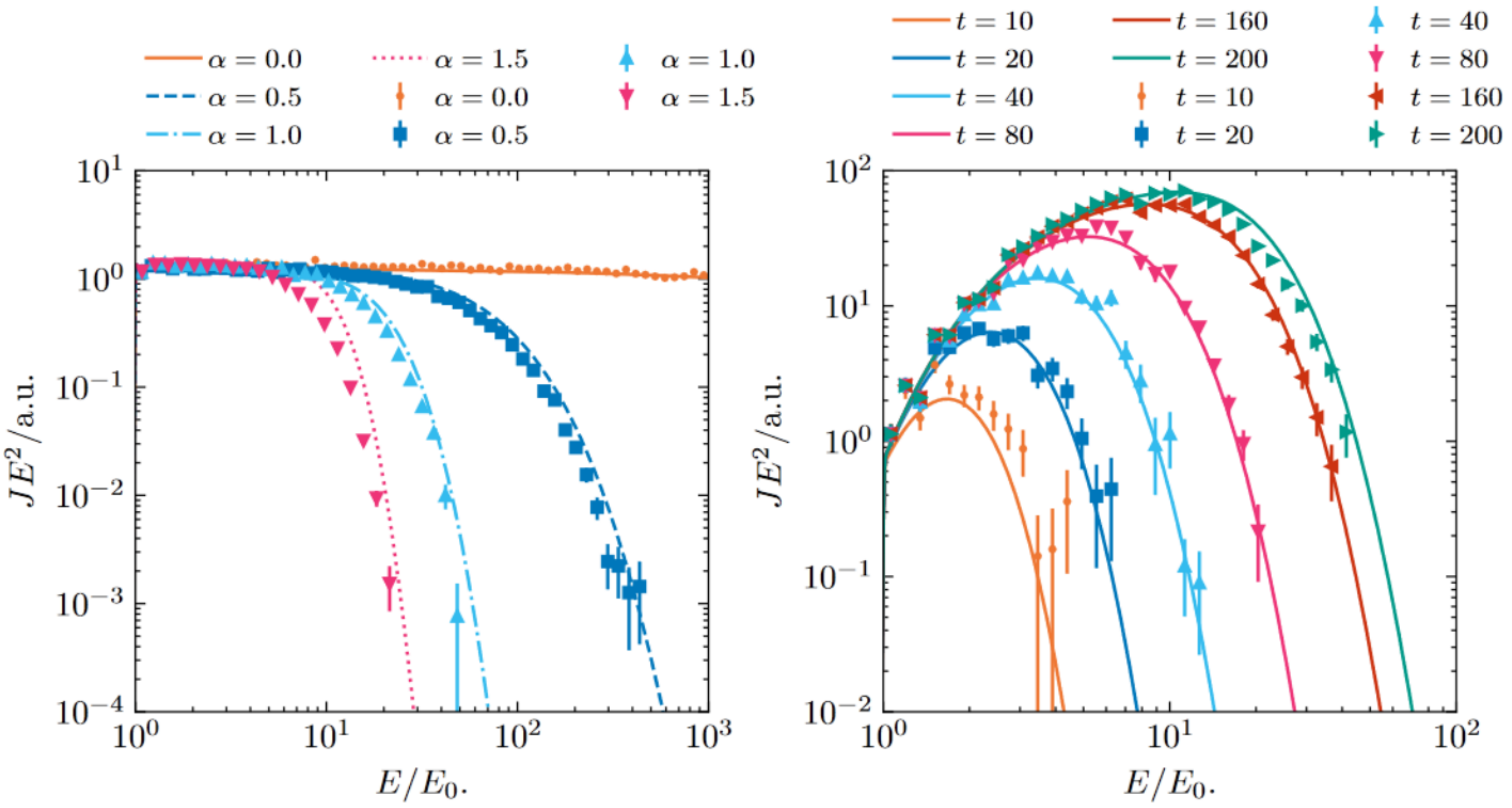




Constant Diffusion Coefficient



Energy Dependent Diffusion



AdiabaticCooling Module

- Follows the original CRPropa approach:
    - Losses are treated independent of transport

- Works with implemented AdvectionFields
    - **Only if**, *getDivergence() is provided*

- No drift terms included or provided

$$\boxed{p_{n+1} - p_n = -\frac{p}{3}(\nabla \cdot \vec{u}) \cdot h}$$

RUB