MODELING SUPERDIFFUSIVE PARTICLE TRANSPORT WITH CRPROPA

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ANOMALOUS DIFFUSION

SUPERDIFFUSION & SUBDIFFUSION

$$\langle(\Delta x)^2\rangle \propto \kappa_\xi t^\xi$$

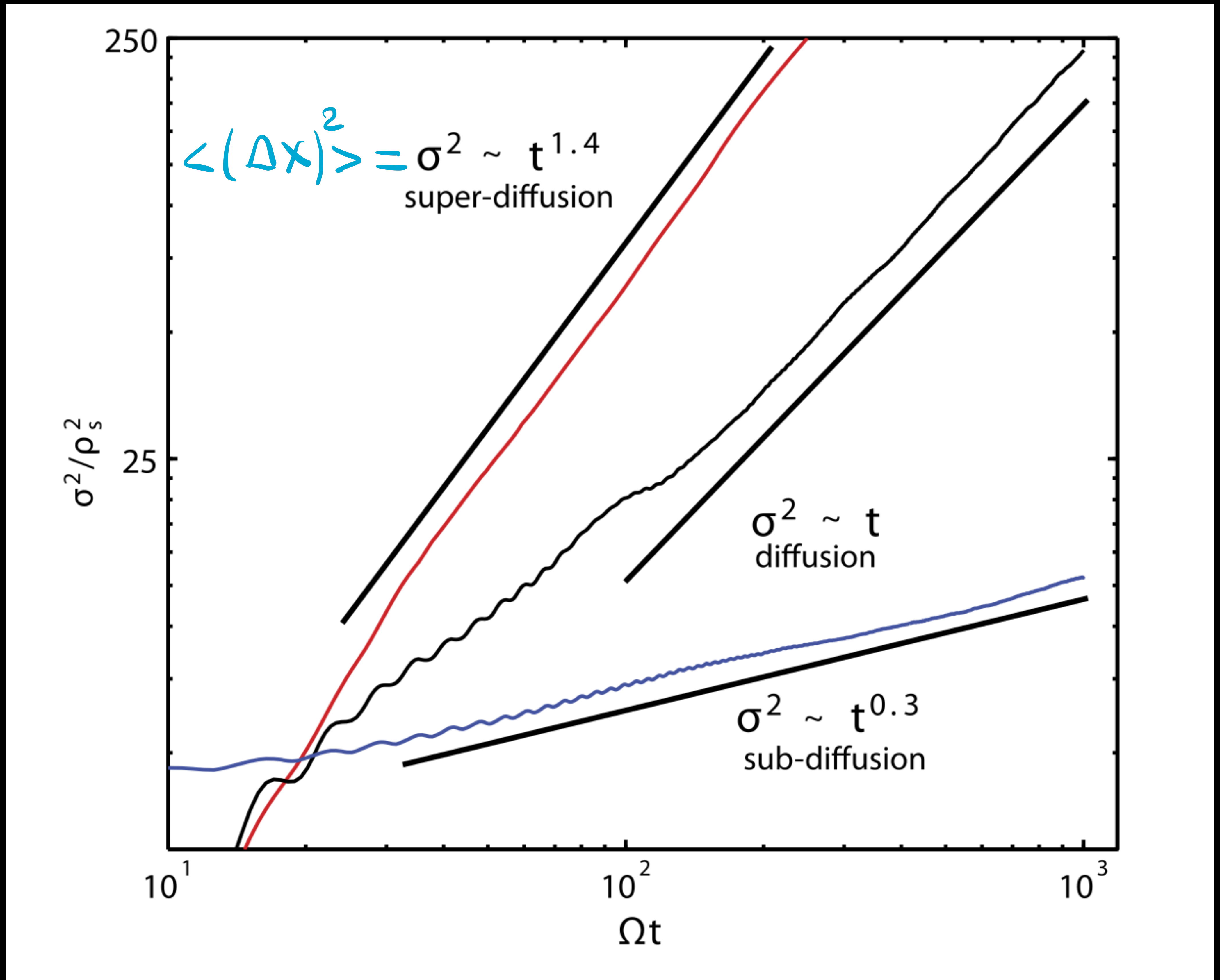
mean square displacement

generalized diffusion coefficient

$0 < \xi < 1$: sub-diffusion

$\xi = 1$: normal (Gaussian) diffusion

$\xi > 1$: super-diffusion



Perrone et al. 2013

SUPERDIFFUSION

EVIDENCE IN THE HELIOSPHERE

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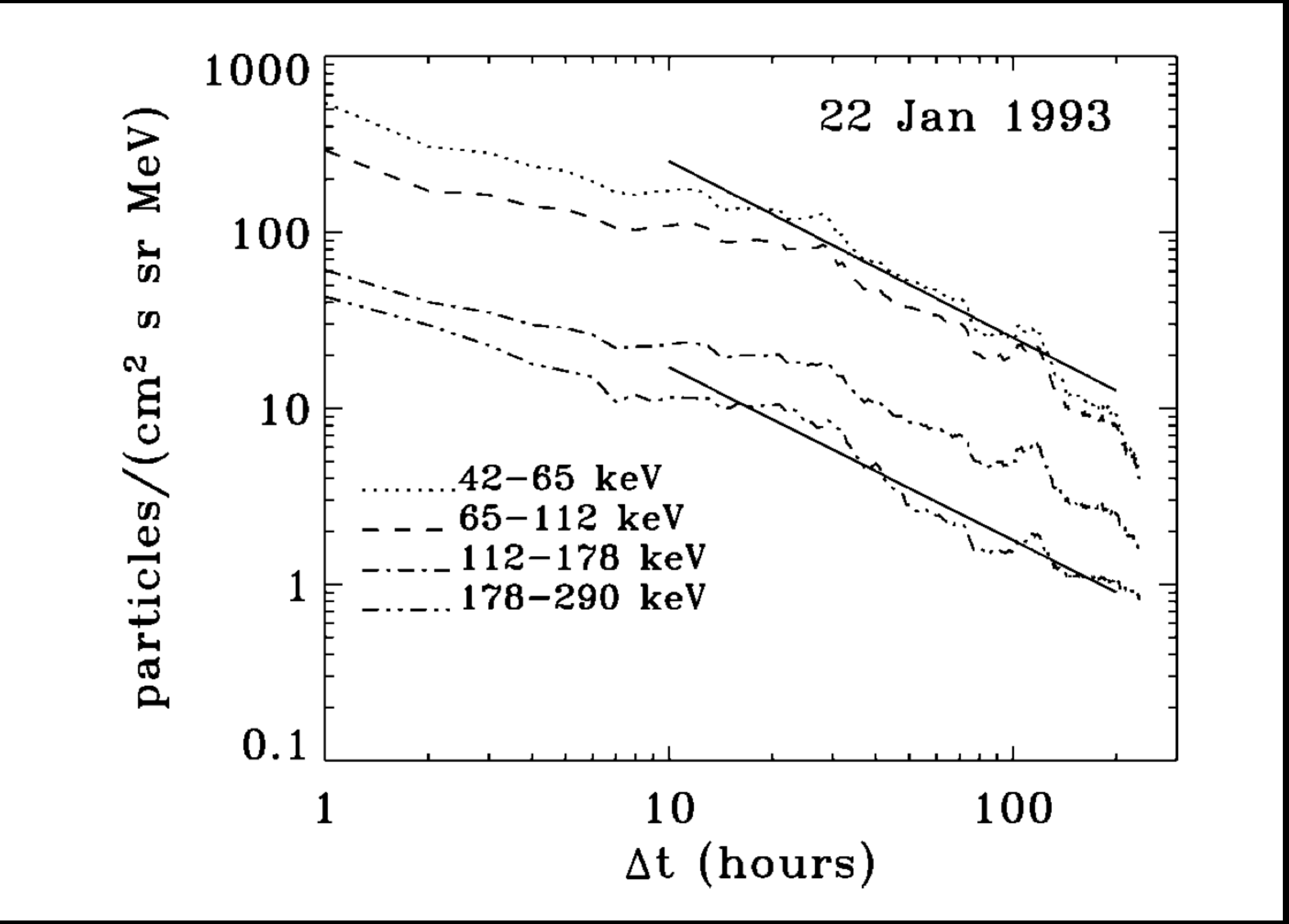
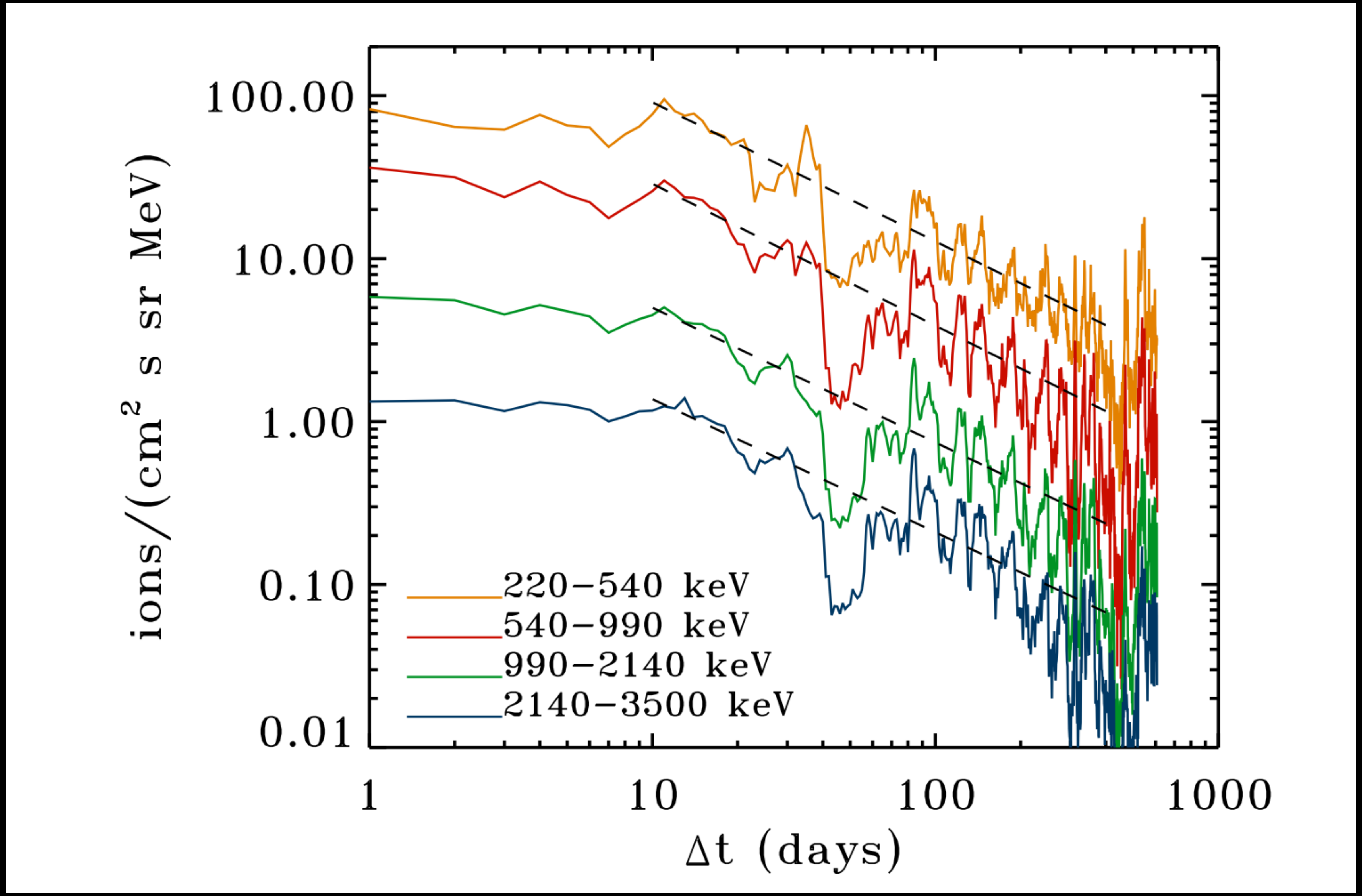


Electron fluxes upstream of interplanetary reverse shock detected by Ulysses, Perri & Zimbardo, 2007



Ion fluxes upstream of solar wind termination shock, Perri & Zimbardo, 2012

SUB- & SUPERDIFFUSION

TIME- & SPACE-FRACTIONAL TRANSPORT EQUATION

$$\frac{\partial f(x,t)}{\partial t} = {}_0D_t^{1-\beta}\left(\frac{\partial}{\partial x}V'(x) + \kappa\nabla^\alpha\right)f(x,t)$$

Riemann-Liouville fractional derivative

$\kappa = \kappa_\xi^{1/\xi}$

Riesz derivative

SUB- & SUPERDIFFUSION

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TIME- & SPACE-FRACTIONAL TRANSPORT EQUATION

$$\frac{\partial f(x,t)}{\partial t} = {}_0D_t^{1-\beta}\left(\frac{\partial}{\partial x}V'(x) + \kappa\nabla^\alpha\right)f(x,t)$$

Riemann-Liouville fractional derivative

$= 1$ for $\beta = 1$

$\kappa = \kappa_\xi^{1/\xi}$

Riesz derivative

$= \nabla^2$ for $\alpha = 2$

SUPERDIFFUSION

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SPACE-FRACTIONAL DIFFUSION-ADVECTION EQUATION

$\beta = 1$

$$\frac{\partial}{\partial t} f(x,t) + a \frac{\partial}{\partial x} f(x,t) = \kappa \frac{\partial^\alpha}{\partial |x|^\alpha} f(x,t) + \delta(x)$$

$[\kappa] = m^\alpha / s$

SUPERDIFFUSION

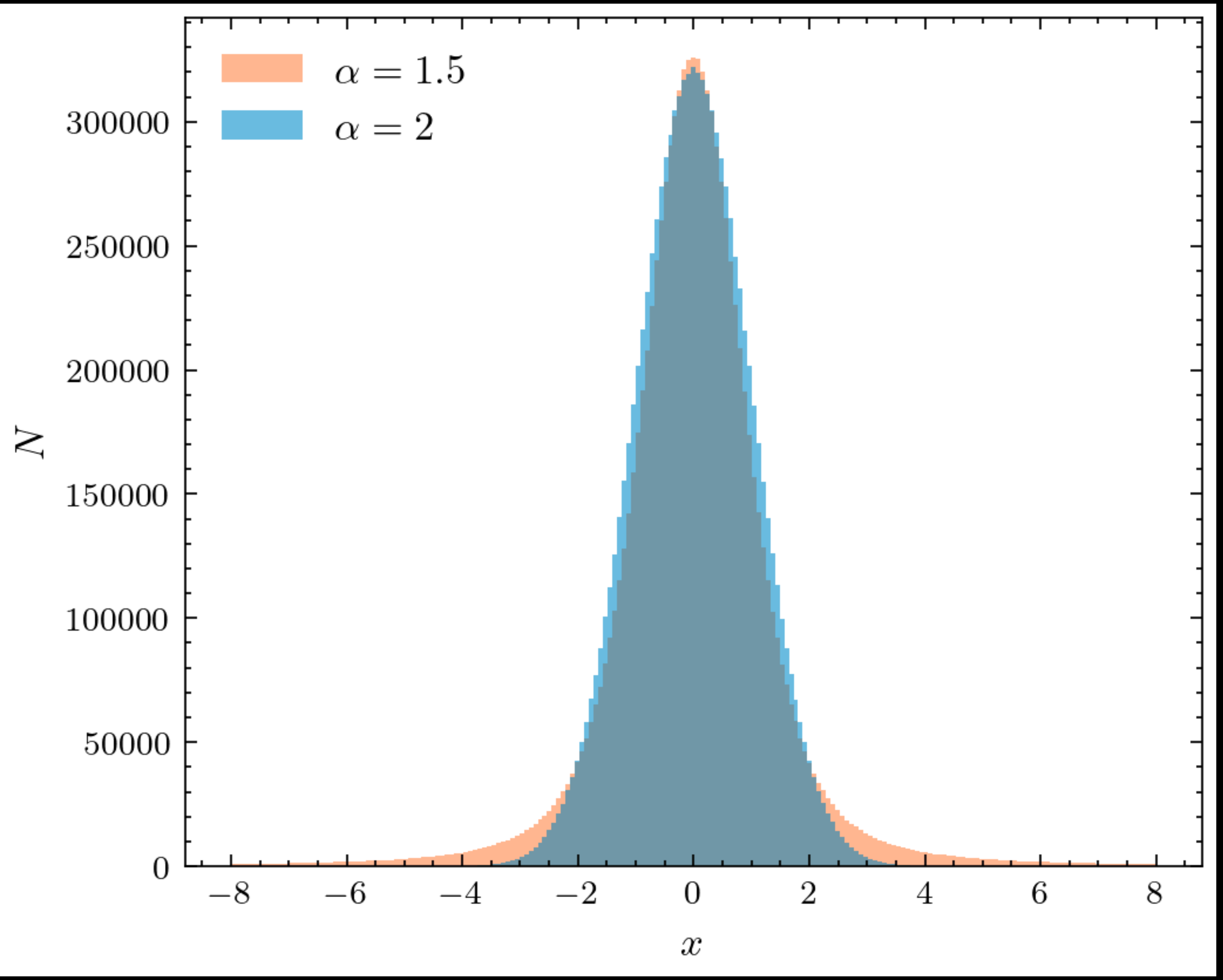
STOCHASTIC DIFFERENTIAL EQUATION: LEVY FLIGHTS

$$\mathrm{d}x = u(x)\mathrm{d}t + \sqrt{2}\kappa^{1/2}\,\mathrm{d}W_t$$

⇓

$$\mathrm{d}x = u(x)\mathrm{d}t + \sqrt{2}\kappa^{1/\alpha}\,\mathrm{d}L_{\alpha,t}$$

- Wiener process $\mathrm{d}W_t \propto \eta_W t^{1/2}$ is exchanged by Lévy process $\mathrm{d}L_\alpha \propto \eta_L t^{1/\alpha}$
- Random numbers η_L are drawn from α-stable Lévy distribution.



Sample of 10^7 random numbers drawn from a α-stable Lévy distribution

SUPERDIFFUSION

STOCHASTIC DIFFERENTIAL EQUATION: LEVY FLIGHTS

LEVY FLIGHTS
IMPLEMENTATION IN CRPROPA

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- Stochastic Differential Equation solved with Euler-Maruyama scheme in *SDESolver*:

$$\vec{x}_{t+1} = \vec{x}_t + A_x \Delta t + B_x \vec{\eta}_x \Delta t^{1/\alpha}$$

- Lévy parameter α is set by ***setLevy()*** function, default is Gaussian ($\alpha = 2$)
- Random numbers $\eta_{x,i}$ are drawn from α-stable Lévy distribution (Chambers, Mellows & Stuck 1976)
- For now, only implemented for spatial diffusion

LEVY FLIGHTS

IMPLEMENTATION IN CRPROPA

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- B_x is given by ***SDEParameter***, e.g. *PureDiffusion, DiffusionAdvection:*

$$B_x = \sqrt{2}\hat{\kappa}^{1/\alpha} = \sqrt{2}\begin{pmatrix} (\kappa_{\|}\epsilon)^{1/\alpha} & 0 & 0 \\ 0 & (\kappa_{\|}\epsilon)^{1/\alpha} & 0 \\ 0 & 0 & \kappa_{\|}^{1/\alpha} \end{pmatrix}$$

- Lévy parameter is considered to be the same in parallel and perpendicular direction to the magnetic field

1. TESTCASE: 1D SUPERDIFFUSION DELTA-INJECTION

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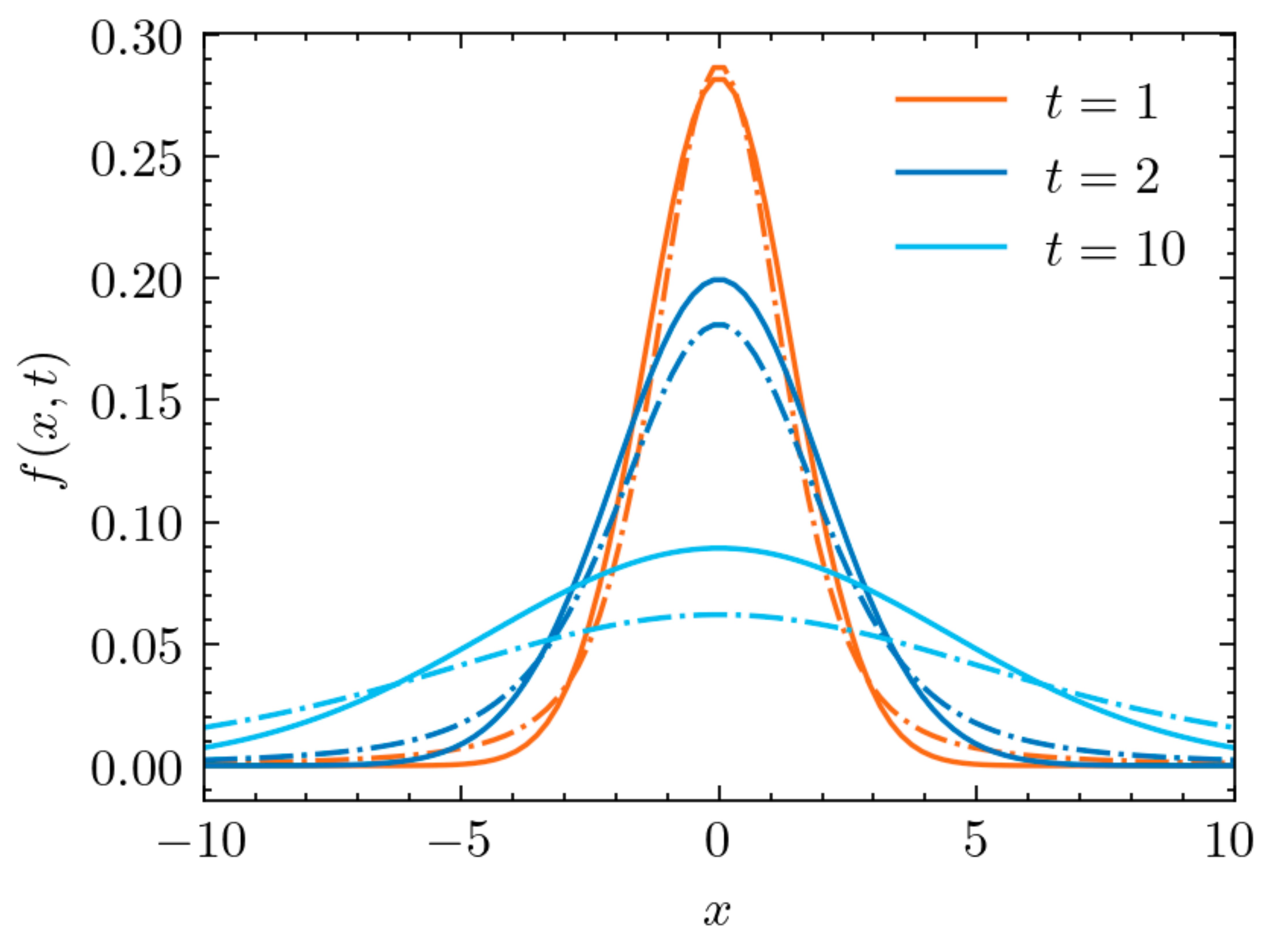
Compare to distribution function $f(x,t)$ approximated by Fourier Series by R. Stern et al. 2013

$$\frac{\partial f}{\partial t} = \kappa \nabla^{\alpha} f, \quad f(x,0) = \delta(x)$$

$$x_{t+1} = x_t + \sqrt{2}\kappa^{1/\alpha}\eta_x \Delta t^{1/\alpha}$$



Gaussian distribution (line) compared to Fourier series approximation for $\alpha = 1.5$ (dashed) over time

1. TESTCASE: 1D SUPERDIFFUSION DELTA-INJECTION

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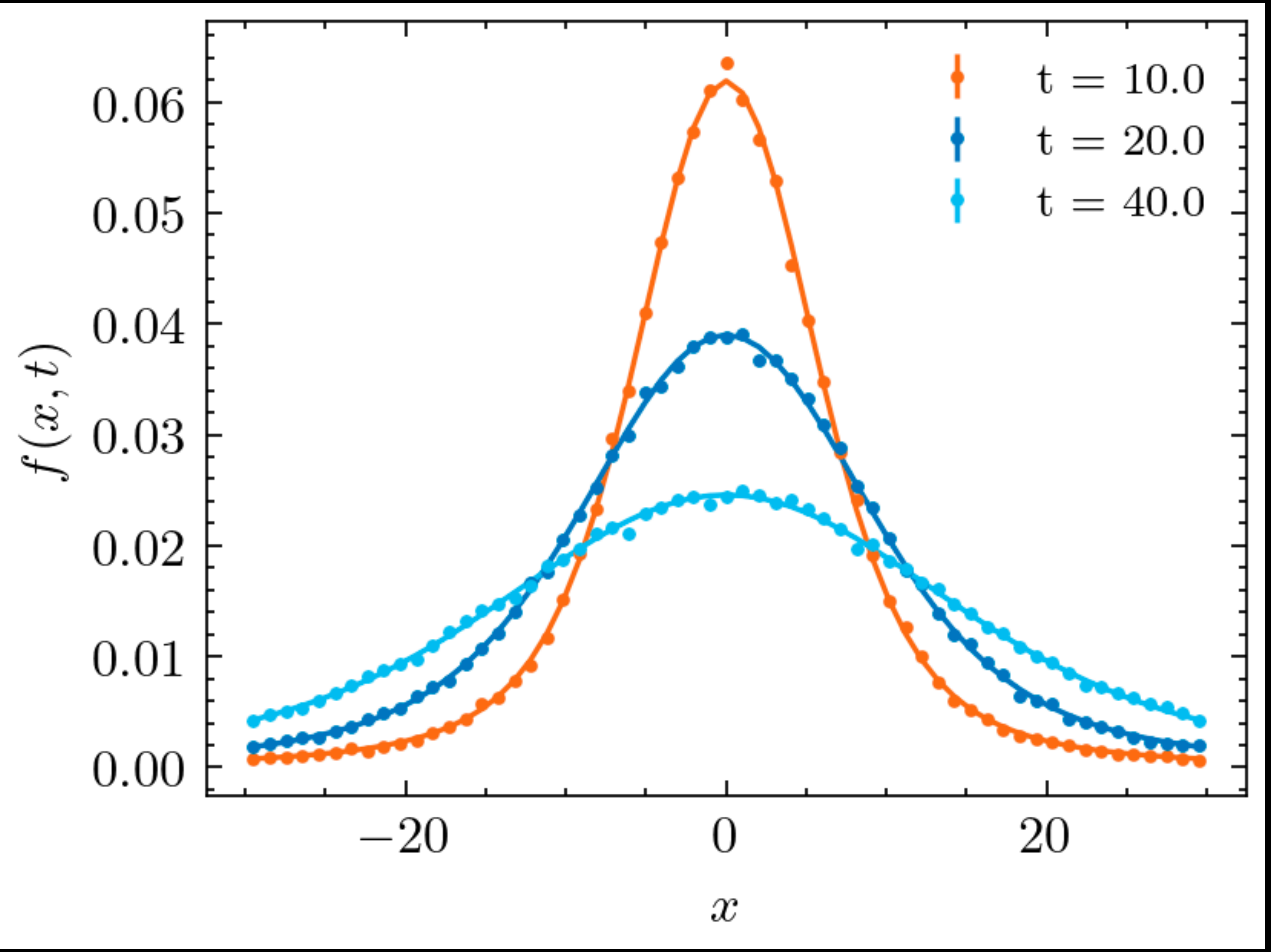
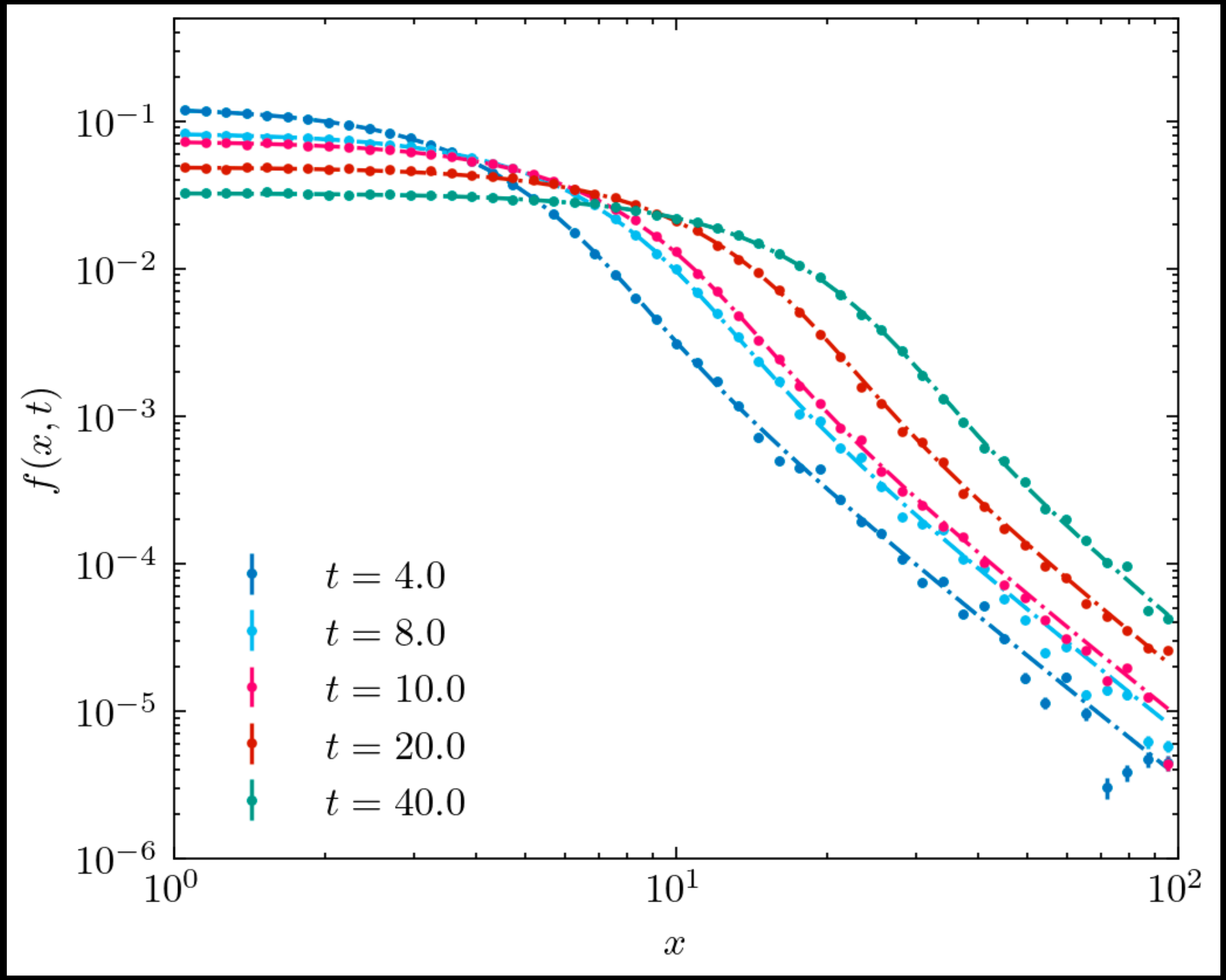


Decaying Lévy distribution



Power-laws in space profile

2. TESTCASE: 1D SUPERDIFFUSION DIFFUSION-ADVECTION AT SHOCK

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Compare to distribution function $f(x, t)$ approximated by Fourier Series by R. Stern et al. 2013

$$\frac{\partial f}{\partial t} = \kappa \nabla^\alpha f + a\frac{\partial f}{\partial x} + \delta(x), \quad f(x,0) = \delta(x)$$

$$x_{t+1} = x_t + a\Delta t + \sqrt{2}\kappa^{1/\alpha}\eta_x \Delta t^{1/\alpha}$$

Construct time-dependent solution by summing of pseudo-particles (e.g. Merten et al. 2017)

$$f(x, t) = \sum f_t(x, t)\Delta T$$

DIFFUSION-ADVECTION EQUATION
APPROXIMATE STATIONARY STATE WITH CRPROPA

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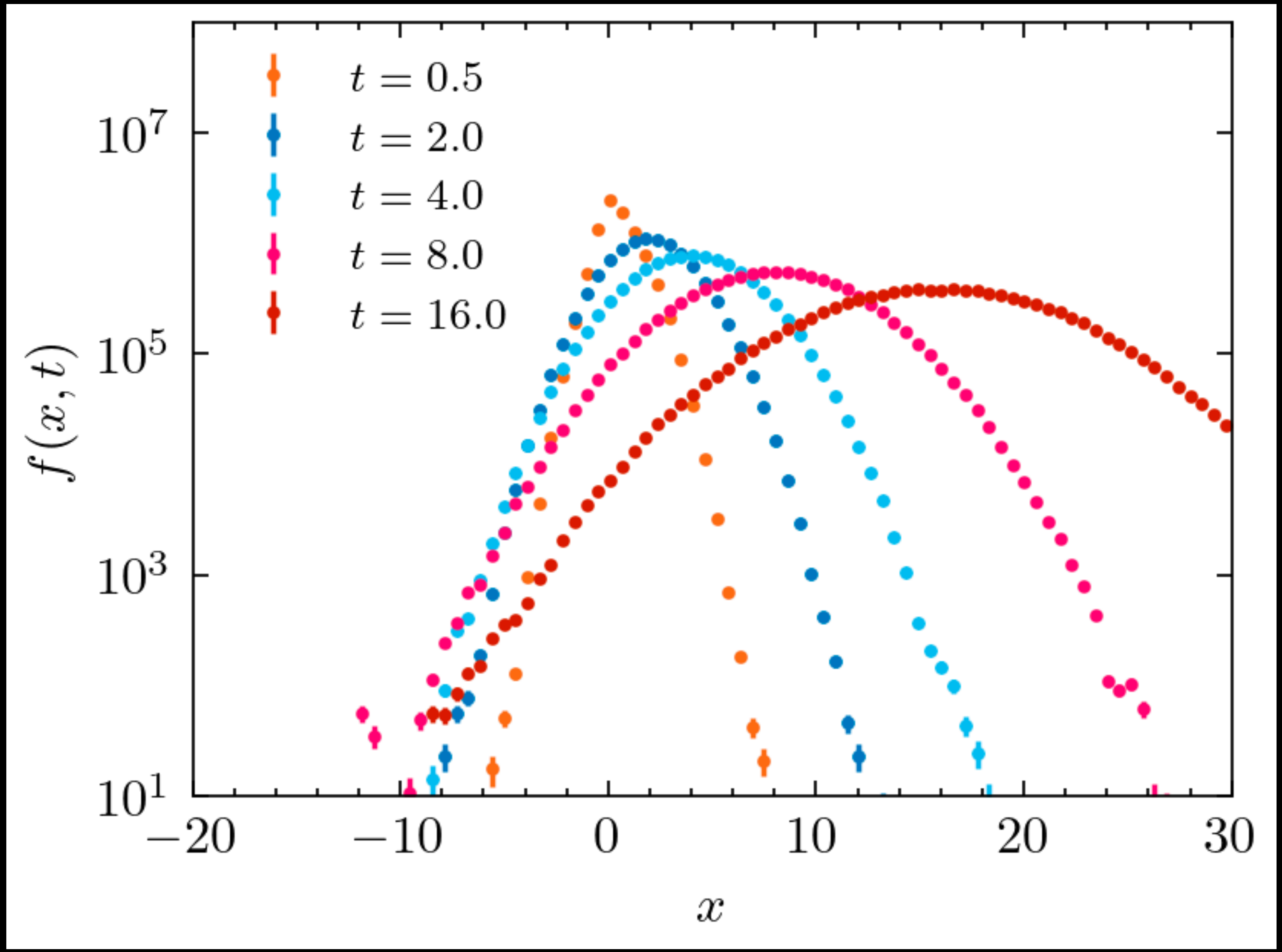
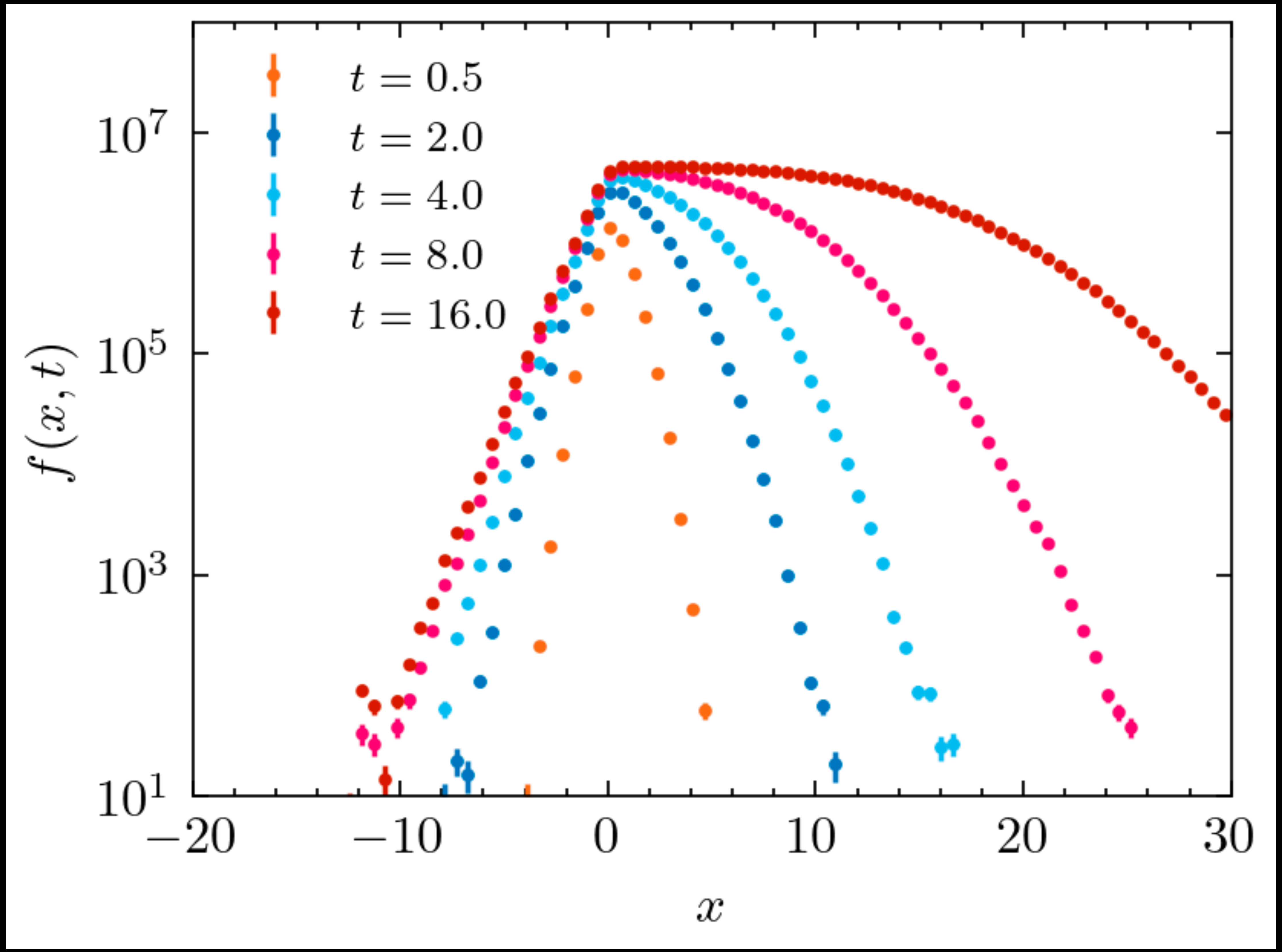


Distribution $f_t(x, t)$ of pseudo-particles at time t



Summed distribution of pseudo-particles $f(x, t) = \sum_i f_t(x, t)\Delta T_i$

2. TESTCASE: 1D SUPERDIFFUSION
DIFFUSION-ADVECTION AT SHOCK

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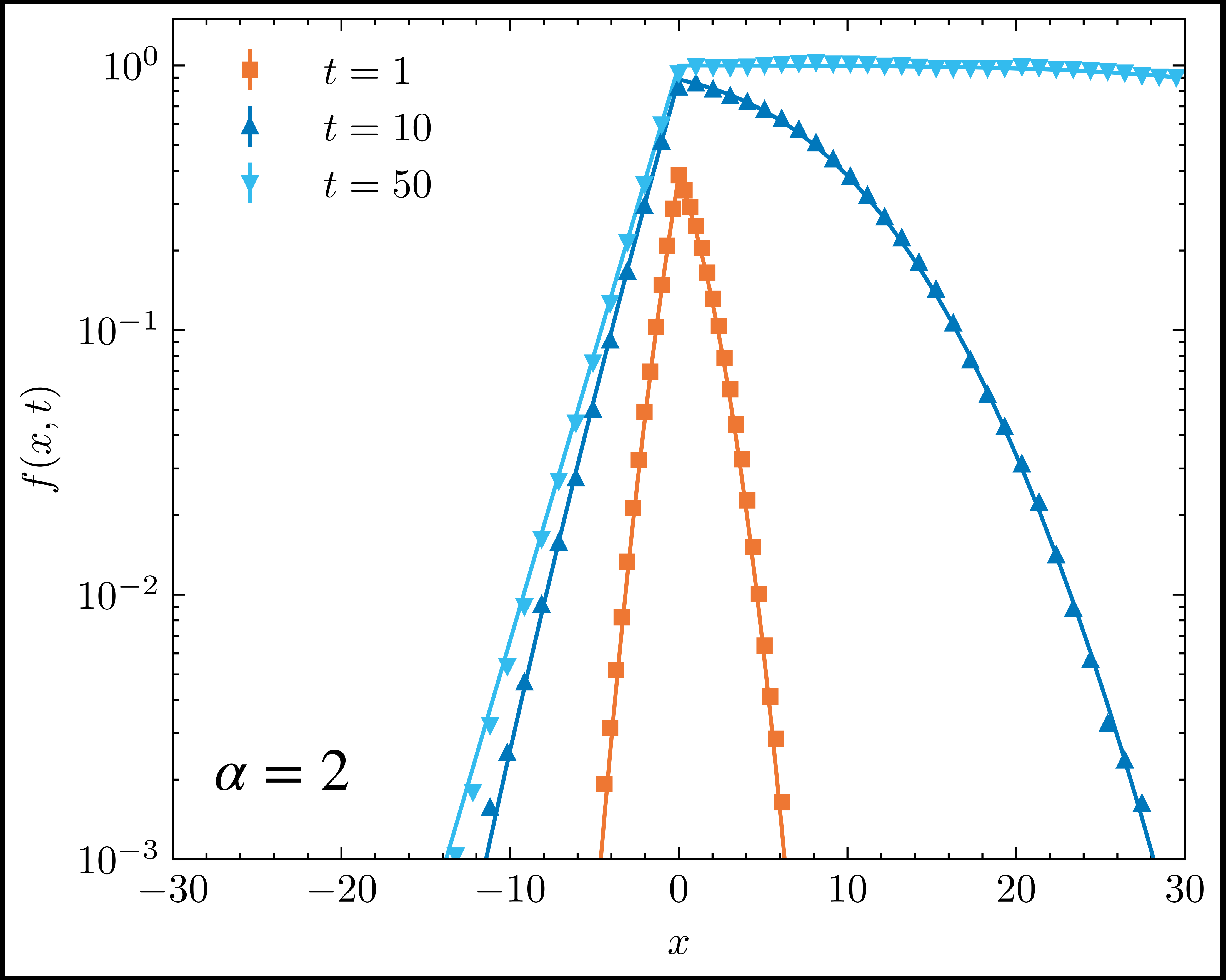

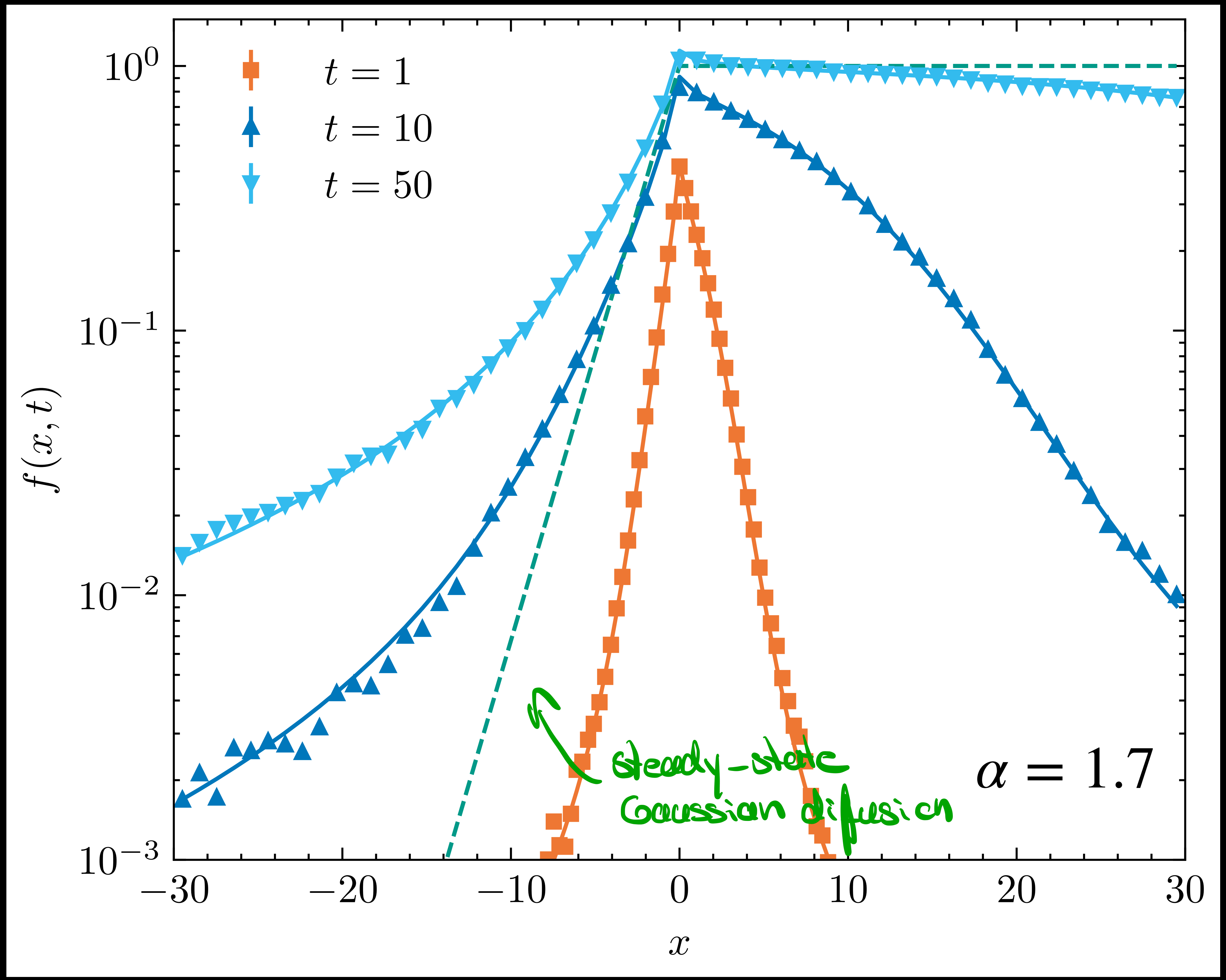

Effenberger et al. (in preparation)

2. TESTCASE: 1D SUPERDIFFUSION
DIFFUSION-ADVECTION AT SHOCK

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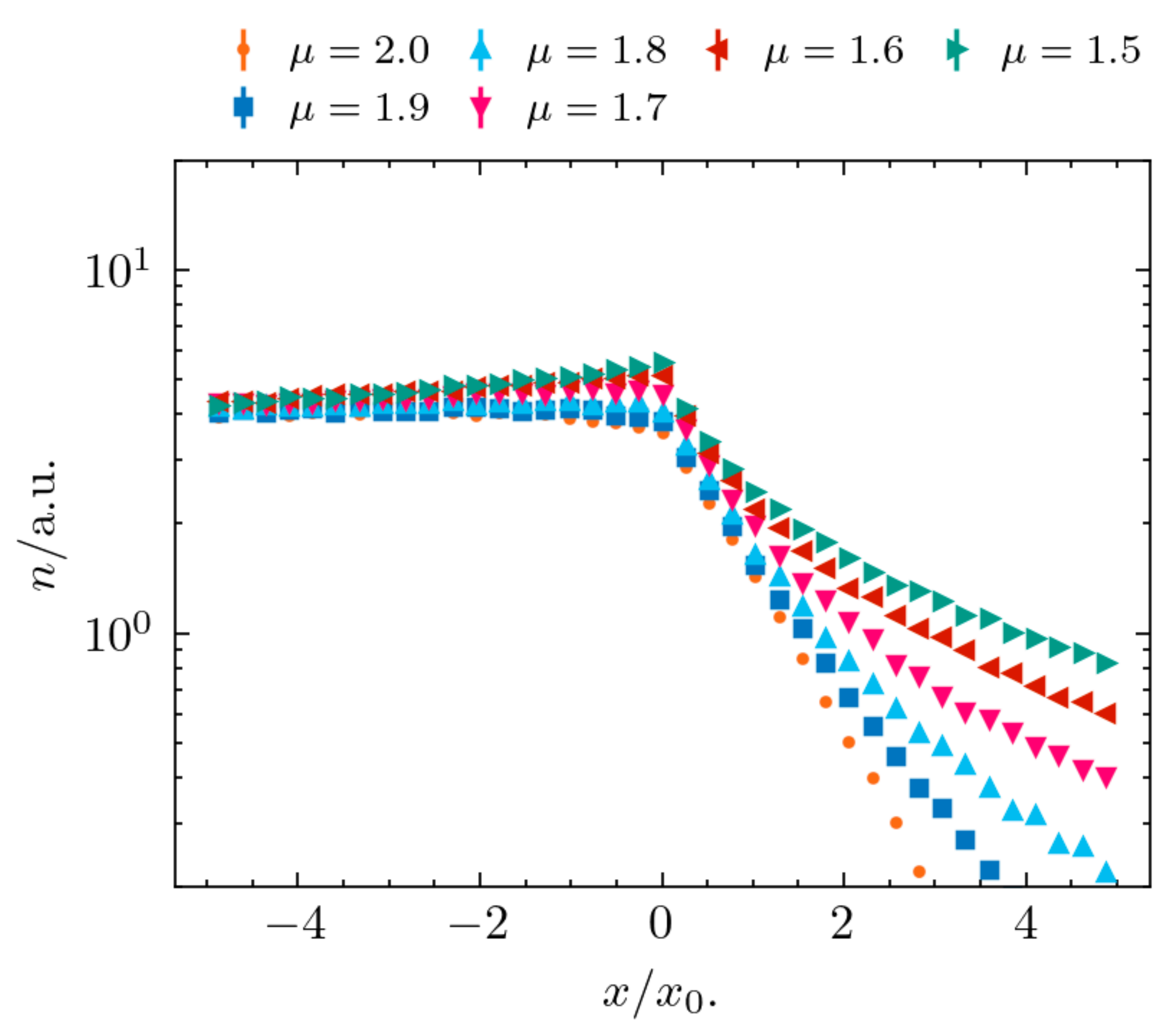
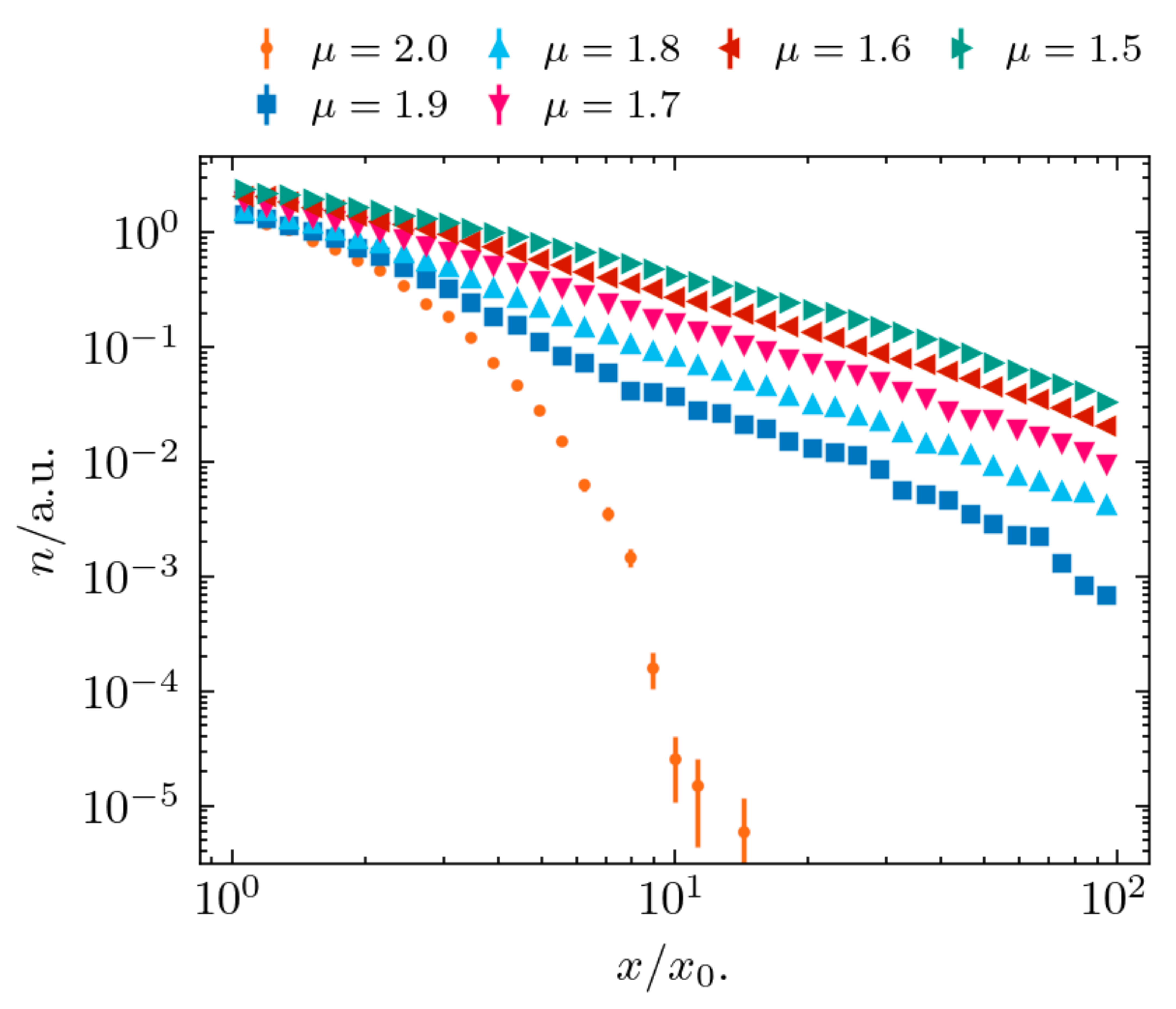


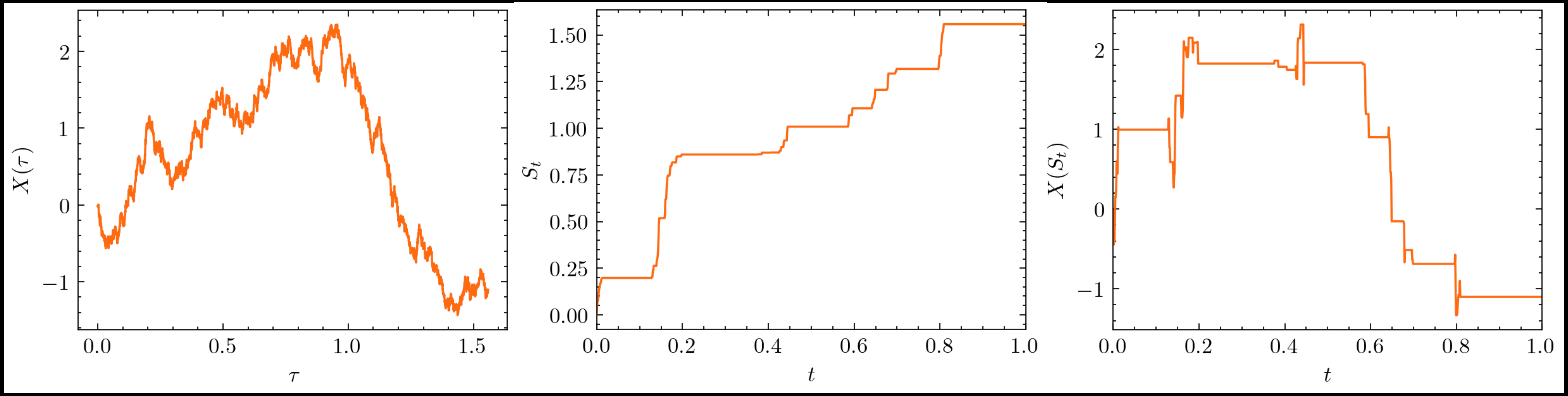


PRELIMINARY: SUBDIFFUSION

$\beta = 0.5$

- Trajectories $X(S_t)$ are obtained by *subordination* from two independent processes: $X(\tau)$, S_t

Magdziarz & Weron, 2007



interpolate $X(S_t)$ from $X(\tau)$

SUMMARY & OUTLOOK

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- Model superdiffuse transport with modified version of CRPropa3.2
  - SDE approach to solve Fractional Fokker-Planck equation
  - Random numbers drawn from the α-stable Lévy distribution
- Delta-injection and diffusion-advection equation fit with Fourier series approximation from Stern et al., 2013
- Future: Distinguish between parallel & perpendicular Lévy parameter
- Future: Investigate subdiffusion & combination of sub- & superdiffusion further

RIEMANN-LIOUVILLE & RIESZ FRACTIONAL DERIVATIVE

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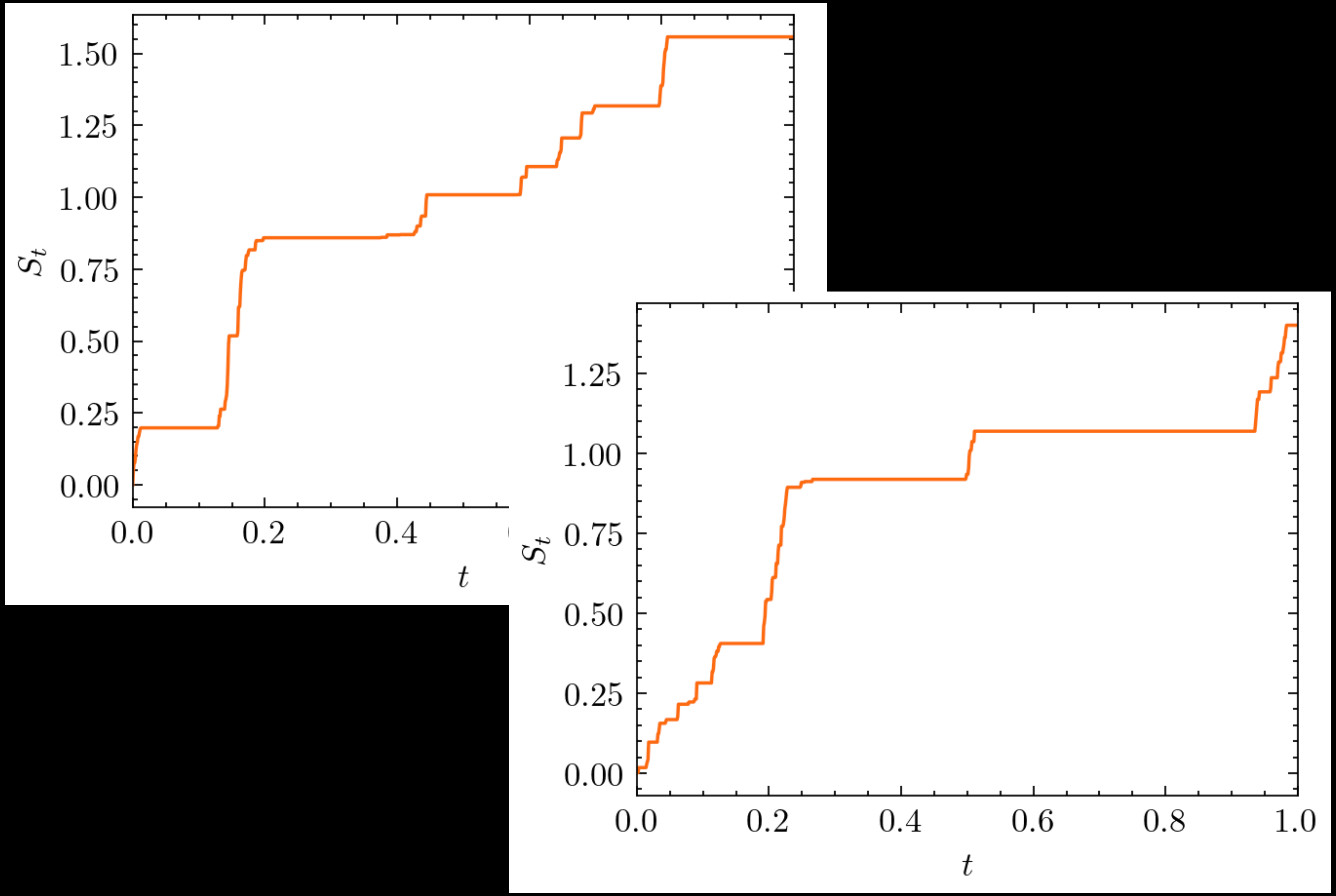
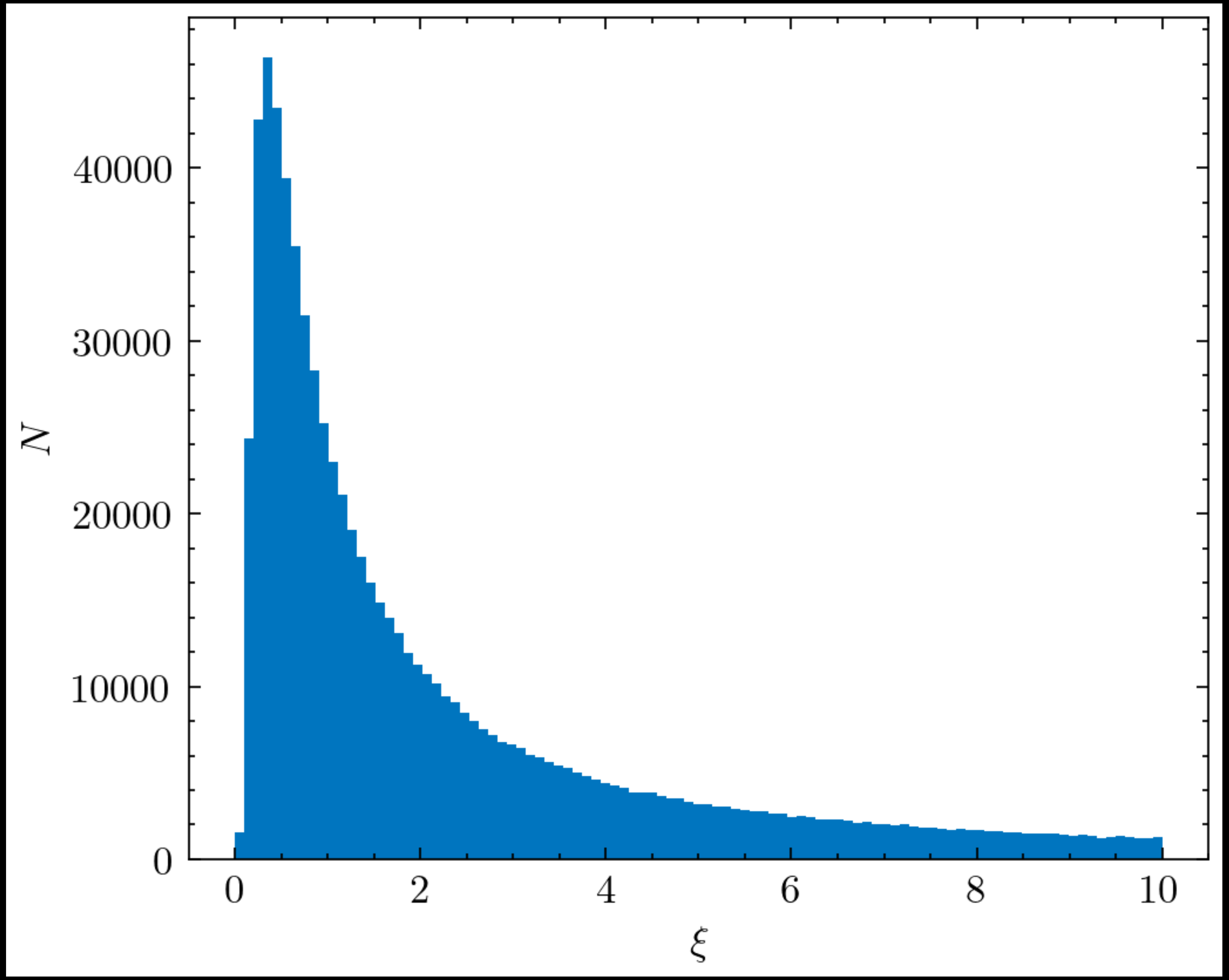
$$ {}_0D_t^{1-\beta} f(t) = \frac{1}{\Gamma(\beta)} \frac{d}{dt} \int_0^t (t-s)^{\beta-1} f(s) ds $$

$$ \nabla^\alpha f(x) = -\frac{1}{2\cos(\alpha\pi/2)} \left({}_{-\infty}D_x^\alpha + {}_xD_{+\infty}^\alpha \right) f(x) $$

SUBORDINATOR S_t

FROM SKEWED α-STABLE LEVY DISTRIBUTION





TEST SUBDIFFUSION

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FROM SKEWED α-STABLE LEVY DISTRIBUTION