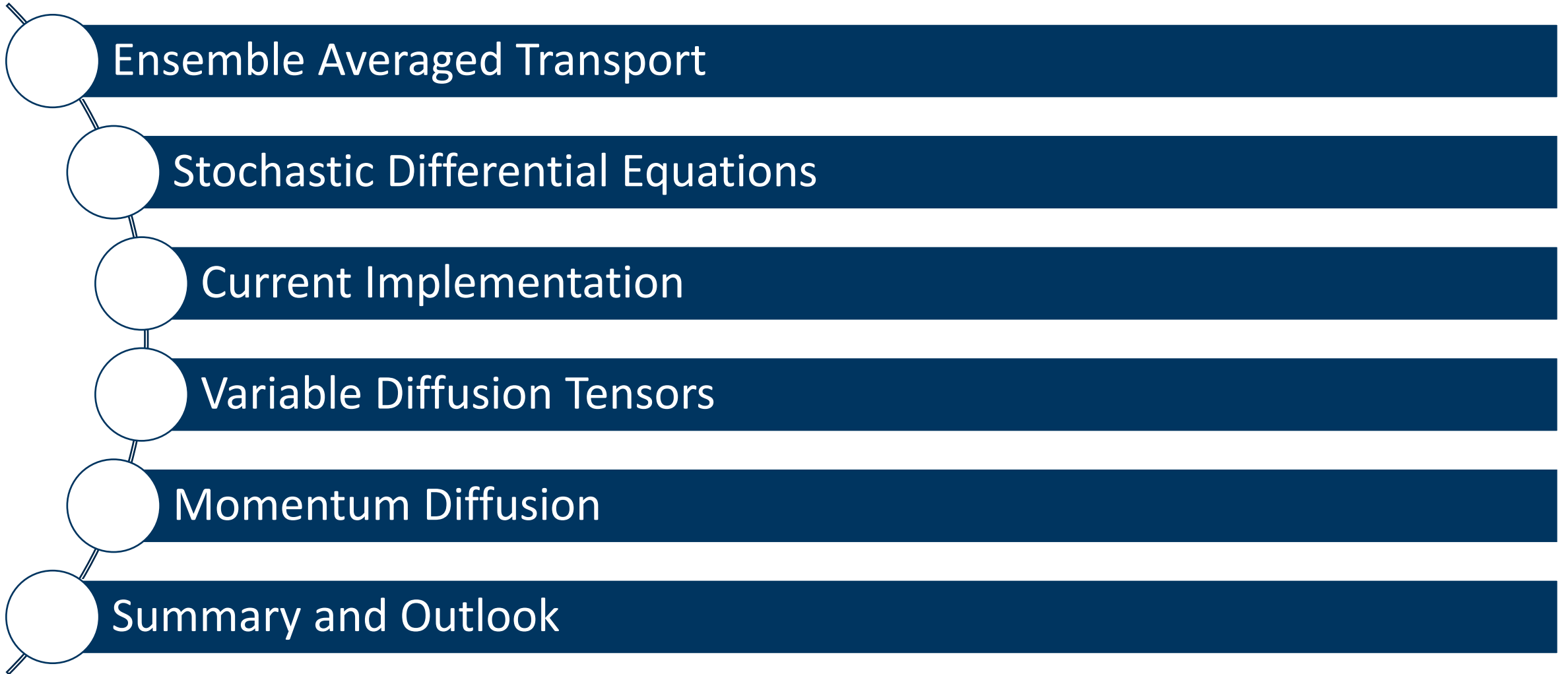


Ensemble Averaged Transport with CRPropa

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Ensemble Averaged Transport

Ensemble Averaged Transport

- Time Evolution of **distribution function**
- Information about the **particle ensemble** but not individual CRs
- Abstract quantities, e.g., **diffusion tensor** need to be provided

Parker Transport Equation

$$\frac{\partial n(\vec{r}, p, t)}{\partial t} + \underbrace{\vec{u} \cdot \nabla n}_{\text{Advection}} = \underbrace{\nabla \cdot (\hat{\kappa} \nabla n)}_{\text{Spatial Diffusion}} + \underbrace{\frac{1}{p^2} \frac{\partial}{\partial p} \left(p^2 \kappa_{pp} \frac{\partial n}{\partial p} \right)}_{\text{Momentum diffusion}} + \underbrace{\frac{p}{3} (\nabla \cdot \vec{u}) \frac{\partial n}{\partial p}}_{\text{Adiabatic Effects}} + \underbrace{S}_{\text{Sources}}$$

Stochastic Differential Equations

Stochastic Differential Equation

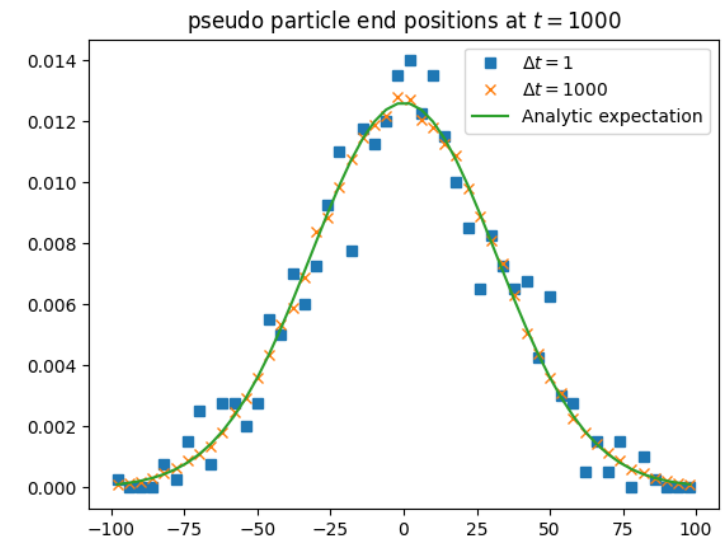
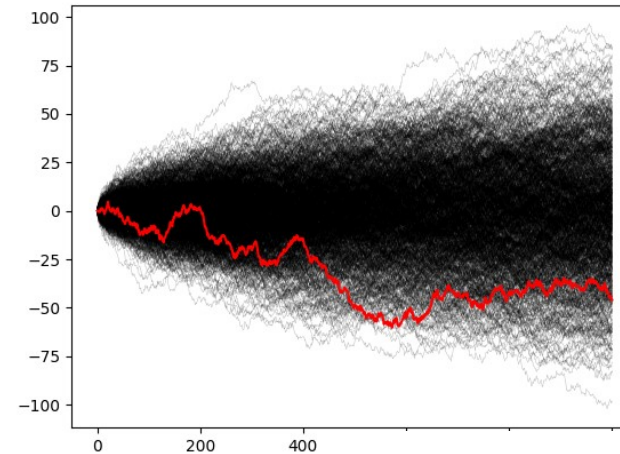
$$dx = A(x, t)dt + B(x, t)dW_t$$

Wiener Process W_t

Properties

1. $W_0 = W(t = 0) = 0$
2. W is time continuous
3. For $s < t \in [0, \infty) \rightarrow W_t - W_s \propto N(0, t - s)$
4. For any $0 \leq s \leq t \leq u < v \in [0, \infty) \rightarrow W_t - W_s$ and $W_v - W_u$ are independent

Trajectories of the standard Wiener process

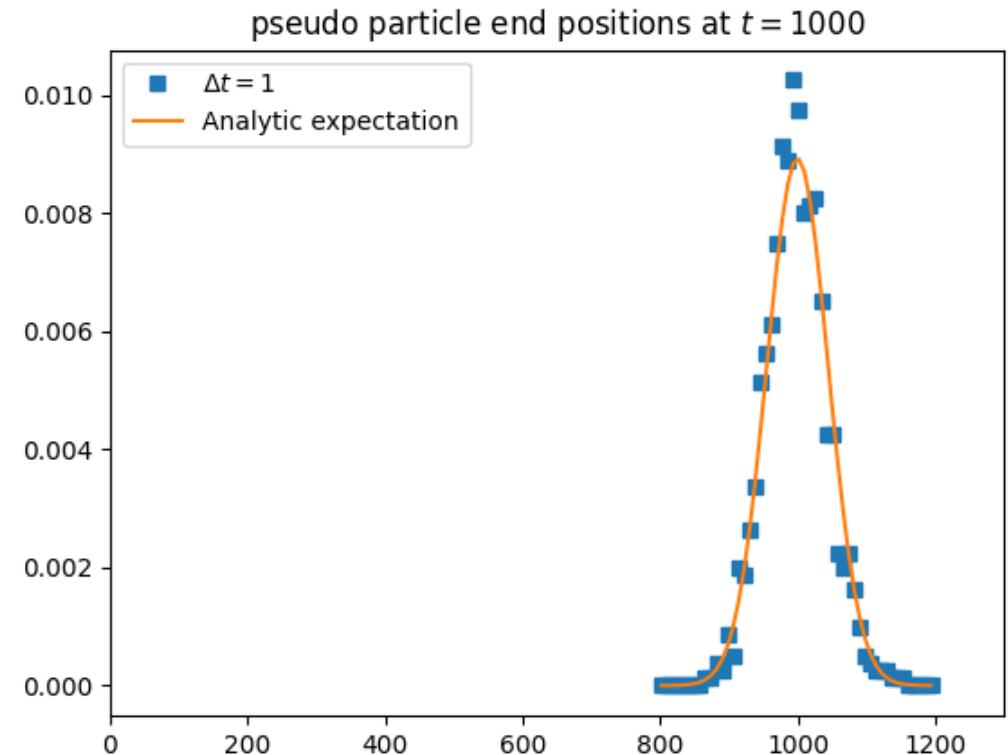


Example – One Dimensional SDE

Drift term and constant diffusion

- $dx = A(x, t)dt + B(x, t)dW_t$
- $A = 1, B = 1$
- $x(t = 0) = 0$
- $T_{\max} = 1000$

Using SDEs to solve a
1-dimensional diffusion advection equation



From Fokker Planck to Stochastic Differential Equation

Fokker Planck Equation:

$$\frac{\partial f(x,t)}{\partial t} = \left[-\frac{\partial}{\partial x} A(x,t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} B(x,t) \right] f(x,t)$$

Can be solved by: Simplification, Discretization, or Transformation

Transformation into Stochastic Differential Equations:

$$dx = A'(x,t)dt + B'(x,t)dW_t$$

Transformation for Coefficients comes from Itô's Lemma

$$A' = A$$

$$B'^2 = B$$

Current Implementation

DiffusionSDE Module

- Arbitrary background magnetic fields
- Anisotropic transport w.r.t. the background field
- Spatially constant Eigenvalues of diffusion tensor κ_{\parallel} and κ_{\perp}
- Energy dependence is a single power law
- Ultra-Relativistic particles: $v = c_0$

$$\vec{x}_{n+1} - \vec{x}_n = (u_x \vec{e}_x + u_y \vec{e}_y + u_z \vec{e}_z) \cdot h + (\sqrt{2\kappa_{\parallel}} \eta_{\parallel} \vec{e}_t + \sqrt{2\kappa_{\perp}} \eta_{\perp,1} \vec{e}_n + \sqrt{2\kappa_{\perp}} \eta_{\perp,2} \vec{e}_b) \cdot \sqrt{h}$$

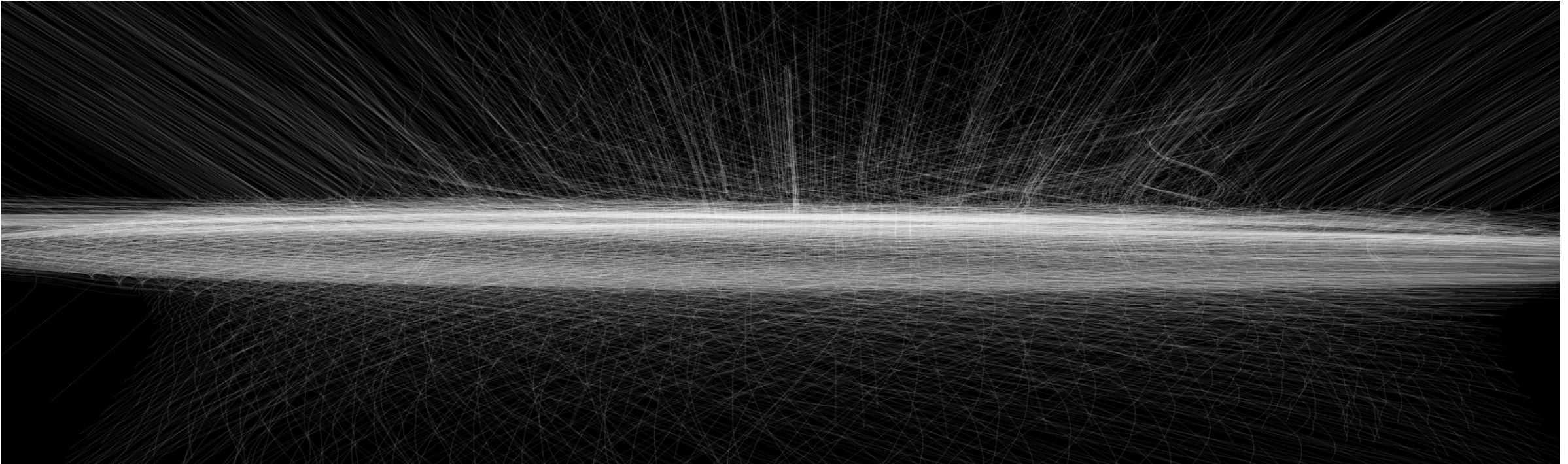
AdiabaticCooling Module

- Follows the original CRPropa approach:
 - Losses are treated independent of transport
- Works with implemented AdvectionFields
 - **Only if**, *getDivergence()* is provided
- No drift terms included or provided

$$p_{n+1} - p_n = -\frac{p}{3} (\nabla \cdot \vec{u}) \cdot h$$

Advantages

- Uses a simple algorithm for the SDE
- No transformation needed by the user
- Works in complicated background fields



Disadvantages

- A lot of code for calculation of local trihedron
 - Computation overhead in simple fields
 - Hard to maintain
- Space dependent Eigenvalues not available
 - No sophisticated shock models
 - No drift terms
- Not easy to extend to anomalous diffusion

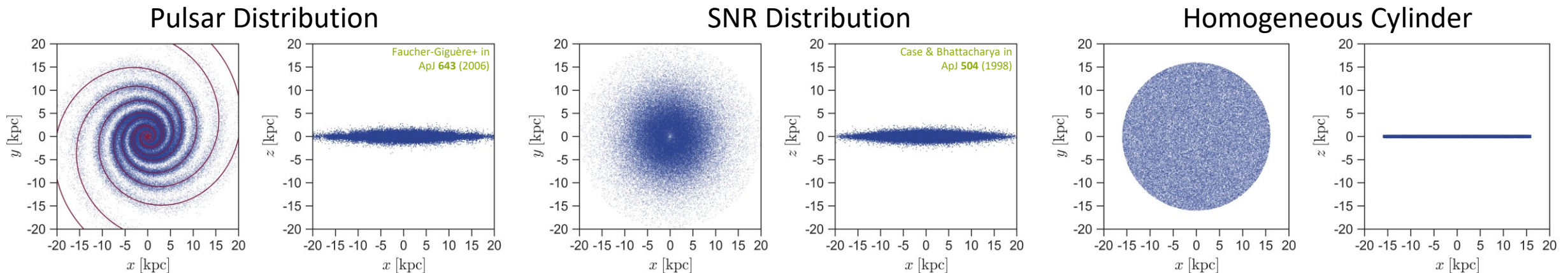
Examples – Transport in the Milky Way

- Source distribution or magnetic field morphology?
- Influence of diffusion ratio ϵ on escape time scales?
- “Classical” scaling of the escape time scale $\tau_{\text{esc}} \propto \kappa^{-1}$?

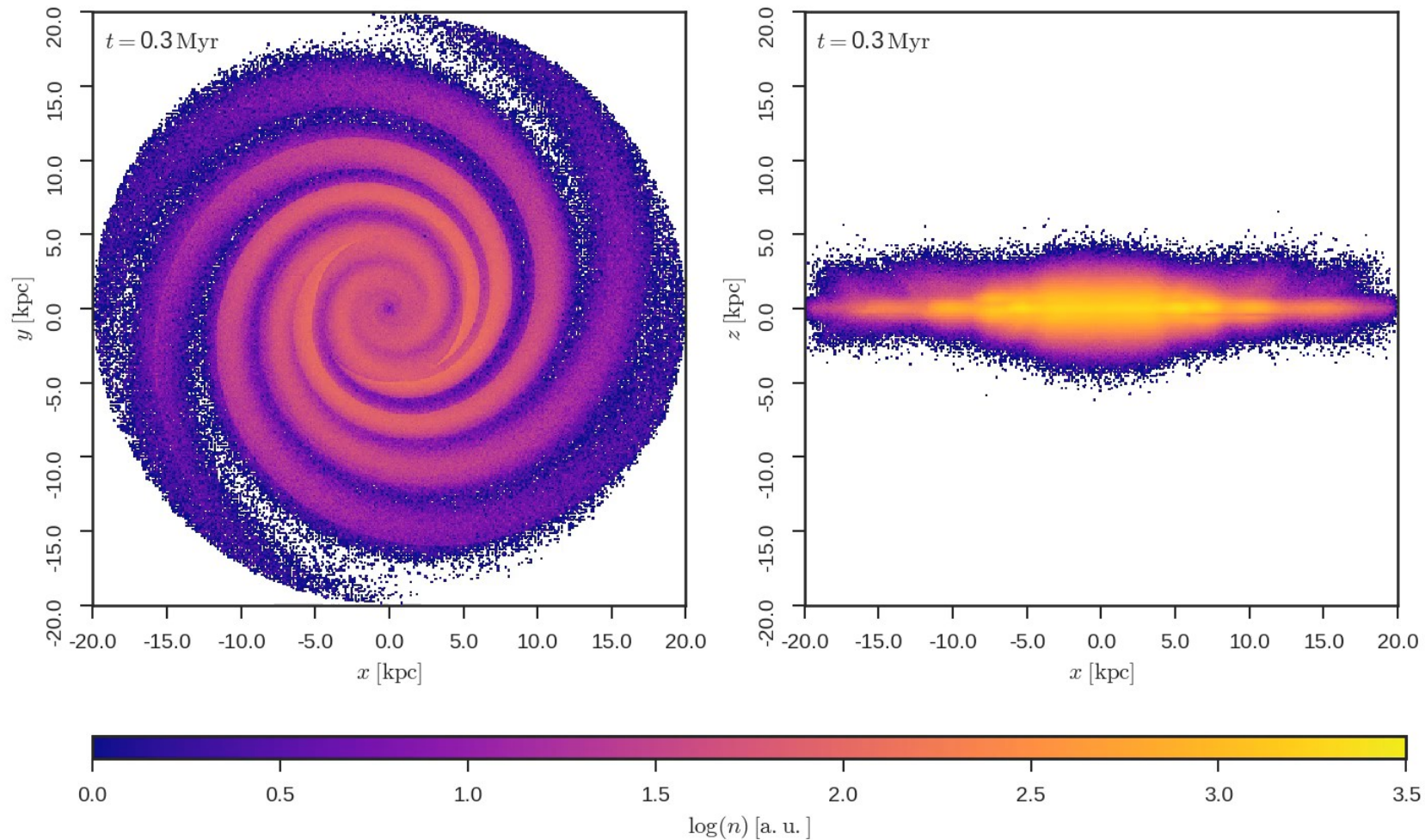
Source Distribution

Compare different source distributions with each other

- Older simulation often assumed a homogeneous cylinder
- Likely source classes (supernova remnants, pulsar wind nebulae, etc.) have a spatial structure
- Burst injection: $S \propto \delta(t - t_0)$
- Injected energies: $E/Z = R \in (10 - 10^5) \text{ TV}$



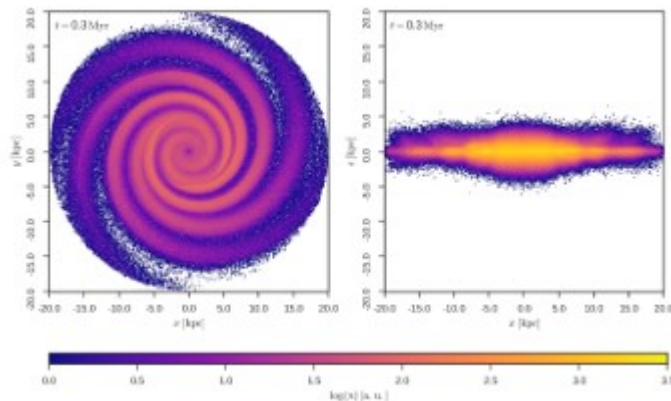
Time Evolution



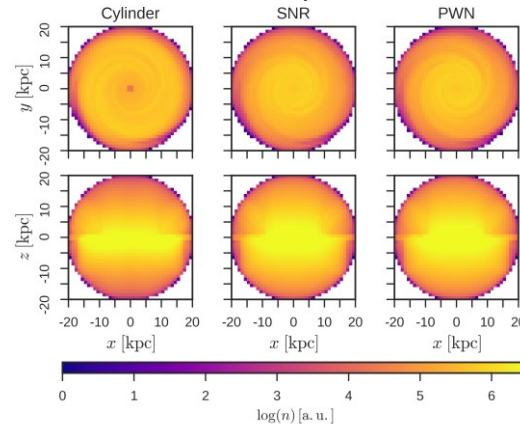
Summary

- Source distribution is relevant on short timescales only.
- Magnetic field morphology plays an important role for the stationary CR density.
- Diffusion ratio determines the magnetic field's influence on CR density.
- Time scales are decreasing with increasing rigidity.

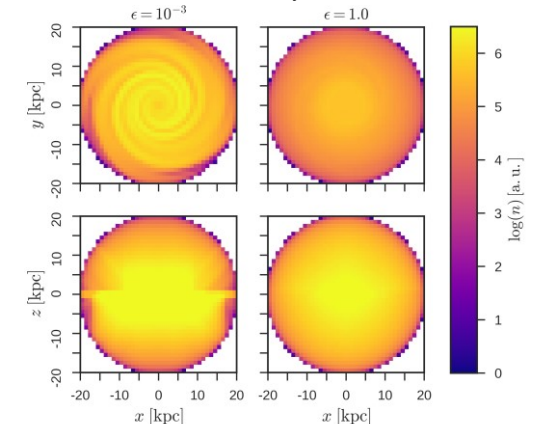
Source: PWN, $R = 10 \text{ TV}$, $\epsilon = 0.01$



$R = 100 \text{ TV}$, $\epsilon = 0.01$



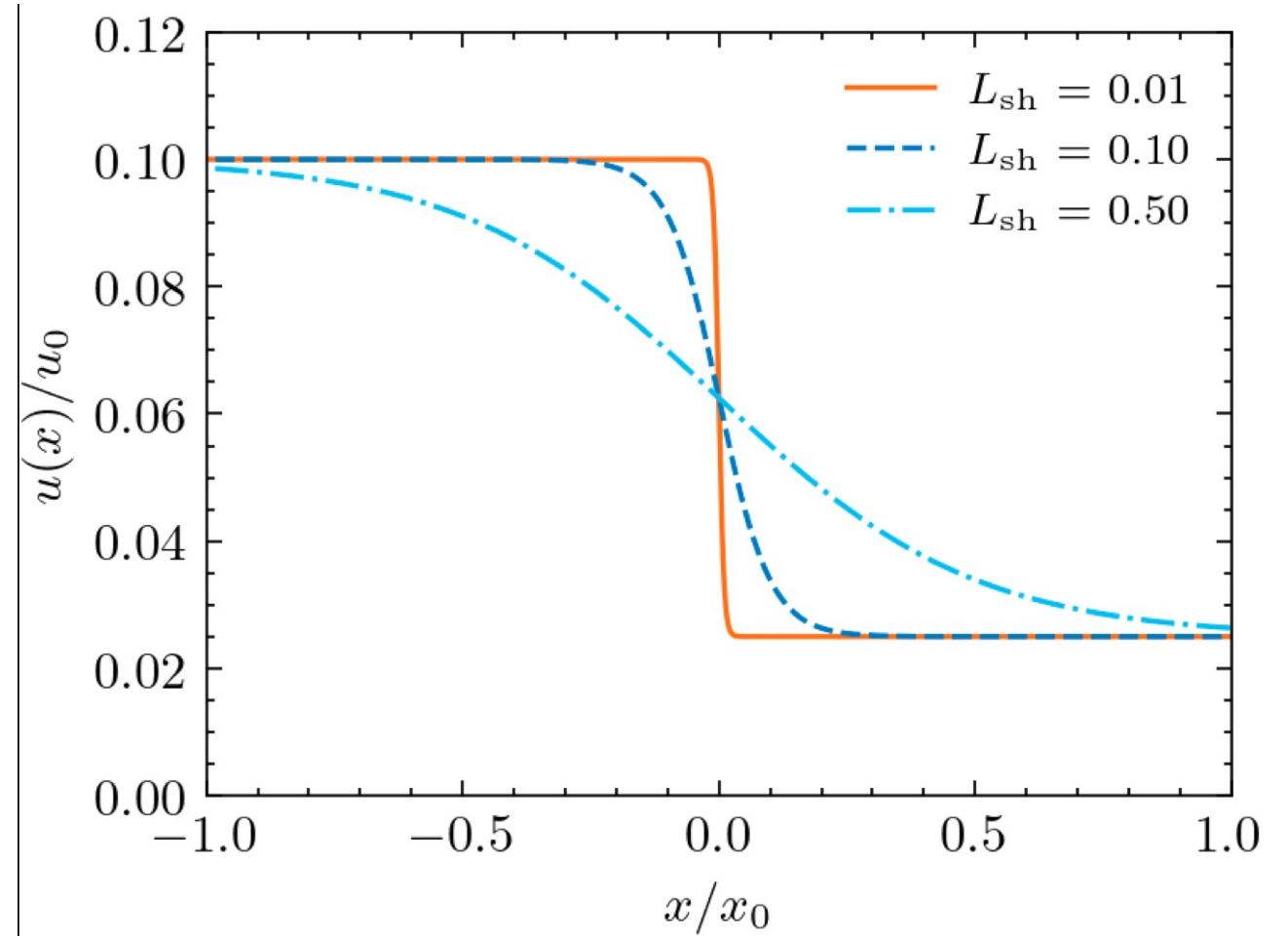
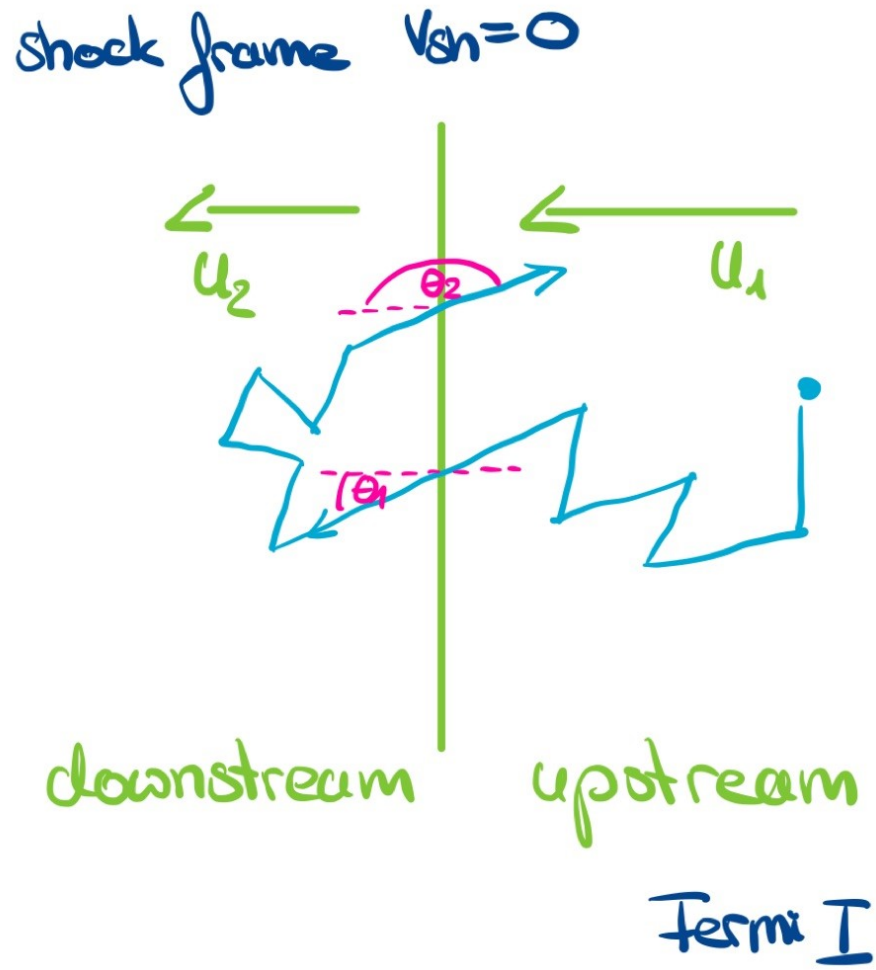
Source: PWN, $R = 10 \text{ TV}$



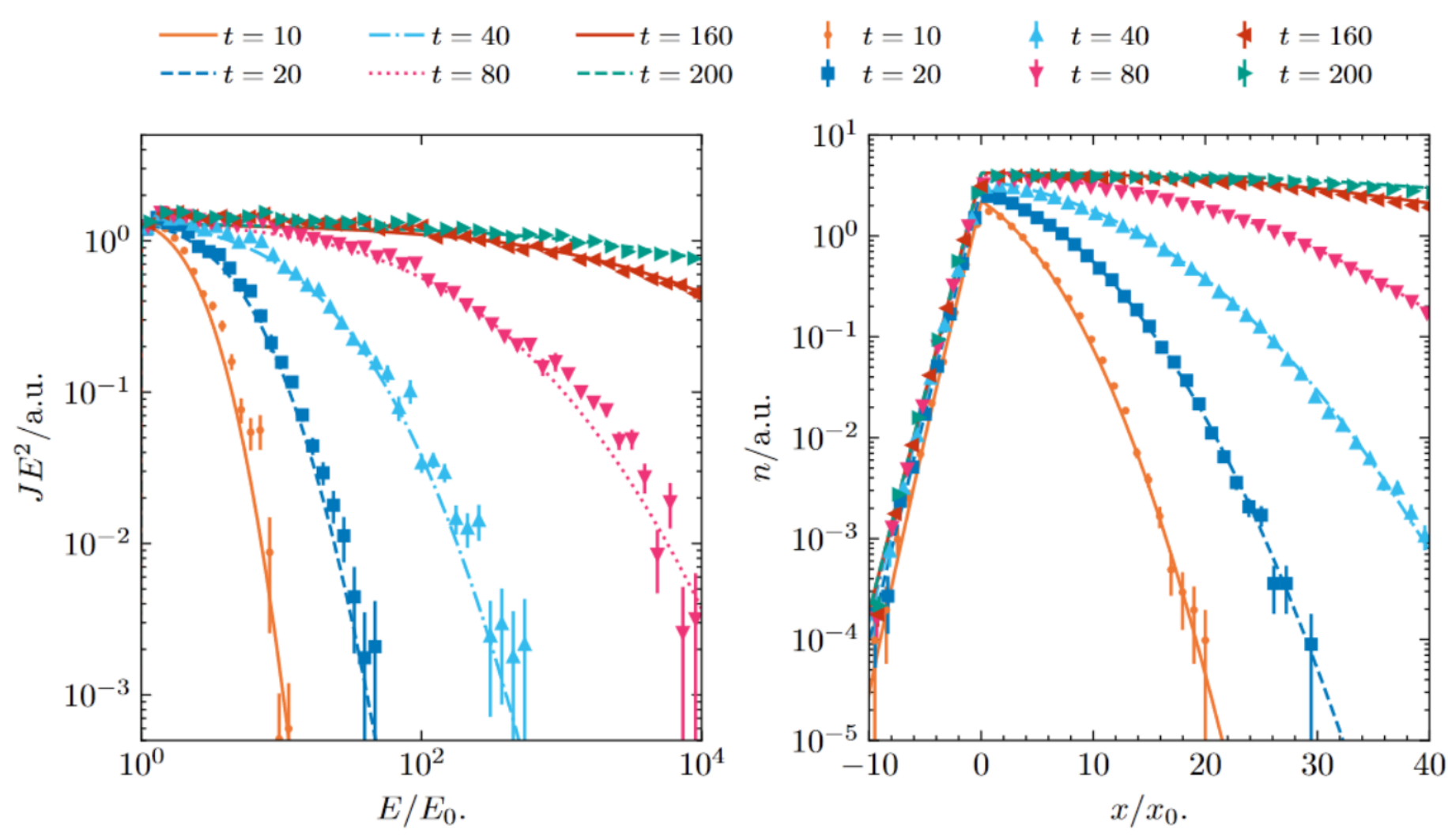
Escape time depends on diffusion ratio and rigidity

$$\tau_{\text{esc}} = (53 \pm 4) \cdot \epsilon^{-0,102 \pm 0,016} \cdot (R/10 \text{ TV})^{0,30 \pm 0,02} \text{ Myr}$$

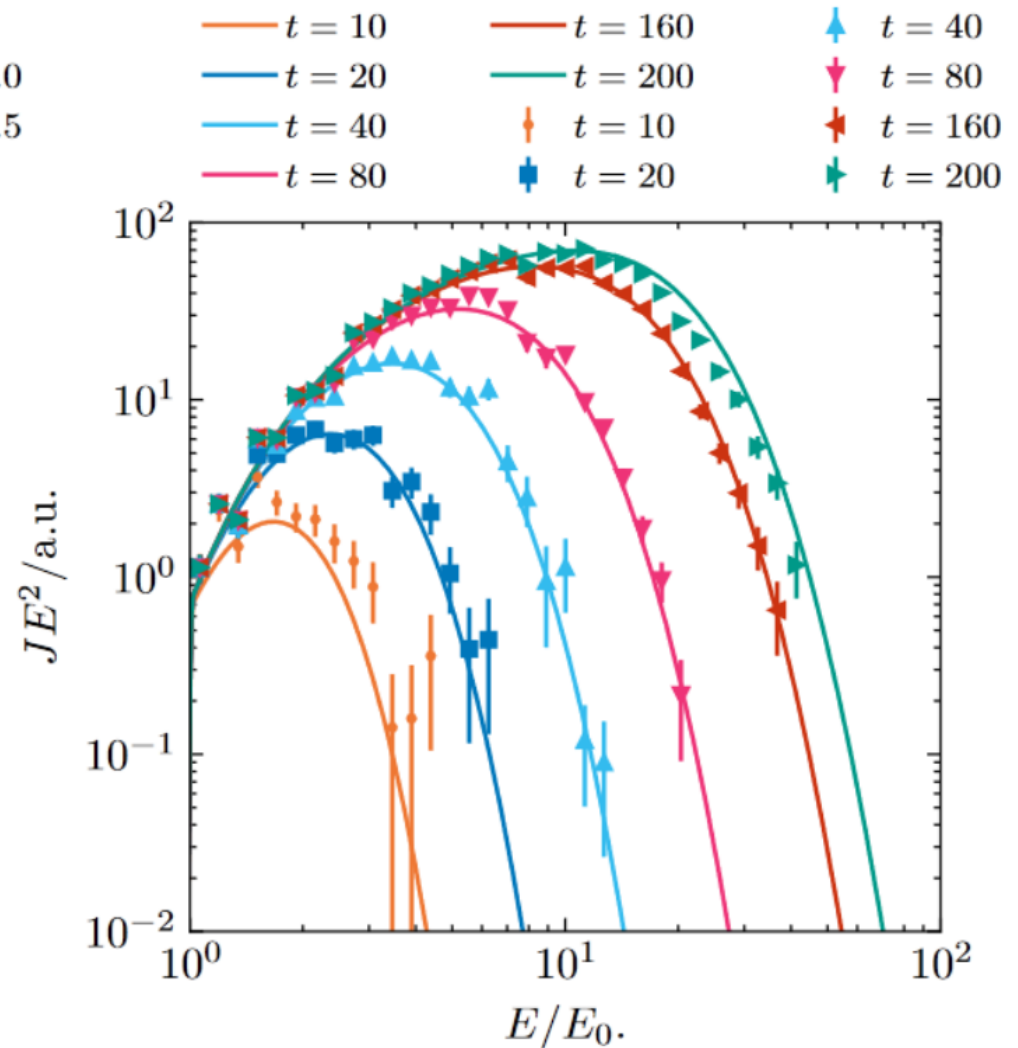
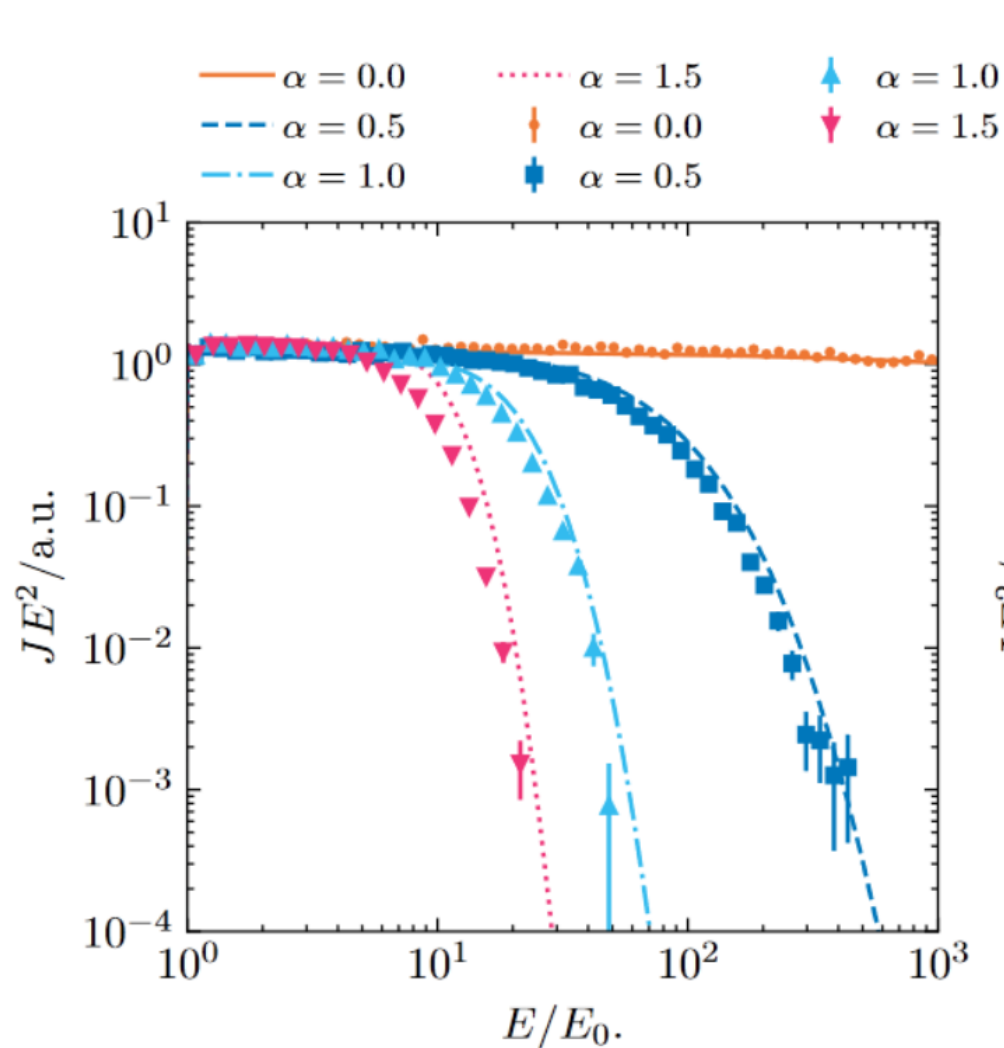
Examples – Diffusive Shock Acceleration



Constant Diffusion Coefficient



Energy Dependent Diffusion



Variable Diffusion Tensors

From FP to SDE

$$\frac{\partial n(\vec{r}, p, t)}{\partial t} + \underbrace{\vec{u} \cdot \nabla n}_{\text{Advection}} = \underbrace{\nabla \cdot (\hat{\kappa} \nabla n)}_{\text{Spatial Diffusion}} + \underbrace{\frac{1}{p^2} \frac{\partial}{\partial p} \left(p^2 \kappa_{pp} \frac{\partial n}{\partial p} \right)}_{\text{Momentum diffusion}} + \underbrace{\frac{p}{3} (\nabla \cdot \vec{u}) \frac{\partial n}{\partial p}}_{\text{Adiabatic Effects}} + \underbrace{S}_{\text{Sources}}$$



Equivalence

$$d\vec{x} = \underline{\vec{u} dt} + \underline{\hat{D} d\vec{w}}$$

$$dp = \underline{-p/3 (\nabla \cdot \vec{u}) dt} + \underline{D_{pp} dw_p}$$

Fokker Planck rewrite of Parker Transport Equ.

$$\frac{\partial n}{\partial t} = \nabla((\hat{\kappa}\nabla n - \vec{v})n) - \vec{w}n + \frac{\partial}{\partial p}(\Omega n) - \left(\frac{\partial}{\partial p}\Omega\right)n + \frac{\partial}{\partial p}\left(D_{pp}p^2\frac{\partial}{\partial p}\left(\frac{n}{p^2}\right)\right) - Ln + S$$

$$\begin{aligned} \frac{\partial n}{\partial t} &= \frac{1}{2}\nabla^2(2\hat{\kappa}n) - \nabla((\nabla\kappa + \vec{u})n) \\ &+ \frac{1}{2}\frac{\partial^2}{\partial p^2}(2Dn) - \frac{\partial}{\partial p}\left(\left(\frac{2}{p}D + \frac{\partial D}{\partial p} - \Omega\right)n\right) \\ &- \left(\frac{\partial\Omega}{\partial p} + L - \nabla\vec{w}\right)n \\ &+ S \end{aligned}$$

What about terms linear in $f(x, t)$?

$$\frac{\partial f(x,t)}{\partial t} = \left[-\frac{\partial}{\partial x} A(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} B(x, t) \right] f(x, t) + C(x, t)f(x, t)$$

Usually included in CR transport equations.

→ Transformation needs to be adapted.

→ Apply weights to the phase space element (*Candidate*)

$$w_0 = \exp(C(x, t)dt)$$

What about constant terms?

$$\frac{\partial f(x,t)}{\partial t} = \left[-\frac{\partial}{\partial x} A(x,t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} B(x,t) \right] f(x,t) + C(x,t)f(x,t) + D(x,t)$$

Usually included in CR transport equations, e.g., source terms

→ Apply additional set of weights to the phase space element (*Candidate*)

$$w_1 = \text{const.}$$

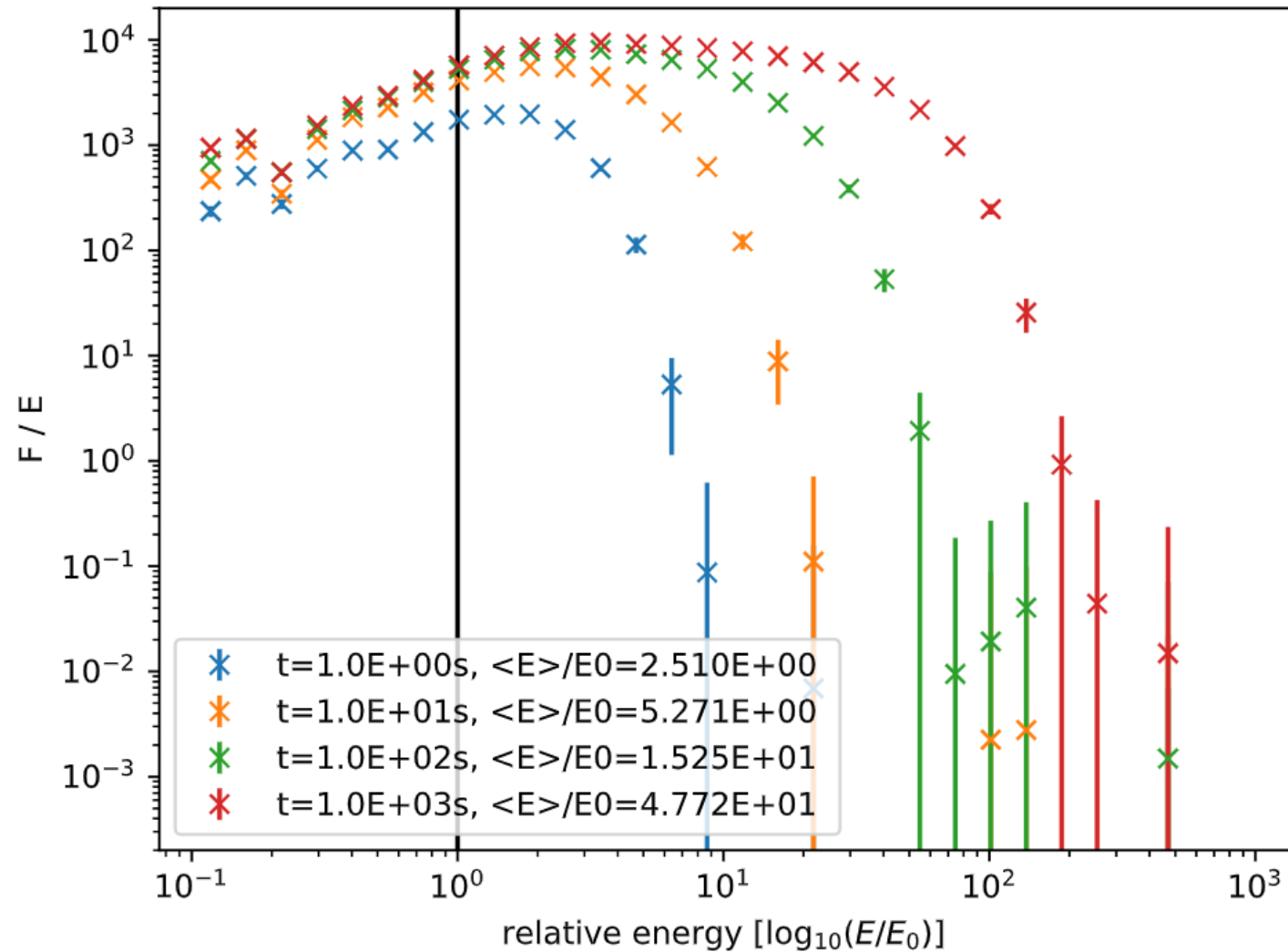
Exact form depends on the setting and introduces physics units, too.

Momentum Diffusion

MomentumDiffusion Module

- Works for constant momentum diffusion coefficient $D_{pp} = \text{const.}$
- Drift terms **not** included
- Separated from the DiffusionSDE module as an additional *energy-change module*
- Currently tested but will be ready for PR soon.

Example – Constant Momentum Diffusion



Summary and Outlook

Summary

- Parker Transport Equation **is not** a Fokker Planck Equation
 - → Has to be rewritten
- SDE becomes more complicated:
 - Drifts
 - Weights
- Current implementation is optimized for
 - Constant diffusion tensors
 - Complex background fields

General SDE Solver

- Re-Usable, e.g., to solve for distribution function instead of density
- Starting from *time forward* SDE
- Unified handling of spatial and momentum diffusion
- Allow for different SDE solver, e.g., second order schemes
 - Might be needed for drifts at shock fronts

Open Questions

- In which frame should the SDE be solved?
- Allow for more complex diffusion tensors?
- Different solvers for magnetic field line integration?

Future Plans

- Implementation and Testing of generalized approach
- Revise Diffusive Shock Acceleration with spatially varying κ
 - Comparison to analytical solutions
- Examine super- and potentially sub-diffusion in more complex settings
 - See also Sophie's talk
- Higher order scheme

Backup